정적분 100題 해설

ver. 20220629

著 : 雀

sukita1729@gmail.com

1. $I=\int_0^{\pi}\frac{x\sin x}{3+\sin^2 x}dx=\int_0^{\pi}\frac{(\pi-x)\sin x}{3+\sin^2 x}dx\ (x\mapsto \pi-x)$

$$2I=\int_0^{\pi}\left(\frac{x\sin x}{3+\sin^2 x}+\frac{(\pi-x)\sin x}{3+\sin^2 x}\right)dx=\int_0^{\pi}\frac{\pi\sin x}{3+\sin^2 x}dx=\int_0^{\pi}\frac{\pi\sin x}{4-\cos^2 x}dx$$

$$=\int_1^{-1}\frac{\pi}{u^2-4}du\ (u=\cos x,\ du=-\sin x dx)$$

$$=\pi\left[\frac{1}{4}\ln\left|\frac{u-2}{u+2}\right|\right]_1^{-1}=\frac{\pi}{2}\ln 3$$

$\therefore I=\frac{\pi}{4}\ln 3$ ■

2. $I=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}dx$

$$\frac{\pi}{2}-2I=\int_0^{\frac{\pi}{2}}dx-\int_0^{\frac{\pi}{2}}\frac{2\sin x}{\sin x+\cos x}dx=\int_0^{\frac{\pi}{2}}\frac{\cos x-\sin x}{\sin x+\cos x}dx=\left[\ln|\sin x+\cos x|\right]_0^{\frac{\pi}{2}}=0$$

$\therefore I=\frac{\pi}{4}$ ■

3. $I=\int_0^{\frac{\pi}{2}}\frac{\cos^{\sqrt{5}}x}{\sin^{\sqrt{5}}x+\cos^{\sqrt{5}}x}dx=\int_0^{\frac{\pi}{2}}\frac{\sin^{\sqrt{5}}x}{\sin^{\sqrt{5}}x+\cos^{\sqrt{5}}x}dx\ (x\mapsto \frac{\pi}{2}-x)$

$$2I=\int_0^{\frac{\pi}{2}}\left(\frac{\cos^{\sqrt{5}}x}{\sin^{\sqrt{5}}x+\cos^{\sqrt{5}}x}+\frac{\sin^{\sqrt{5}}x}{\sin^{\sqrt{5}}x+\cos^{\sqrt{5}}x}\right)dx=\frac{\pi}{2}$$

$\therefore I=\frac{\pi}{4}$ ■

4. $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x+\cos x}dx = \int_0^{\frac{\pi}{4}} \frac{\sin\left(\frac{\pi}{4}-x\right)}{\sin\left(\frac{\pi}{4}-x\right)+\cos\left(\frac{\pi}{4}-x\right)}dx \ \left(x \mapsto \frac{\pi}{4}-x\right)$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\cos x-\sin x}{\cos x}dx = \frac{1}{2}\int_0^{\frac{\pi}{4}}(1-\tan x)dx = \frac{1}{2}\left[x+\ln|\cos x|\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8}-\frac{1}{4}\ln 2$ ■

5. $t=\sqrt[3]{x}$, $dx=3t^2dt$

$\int_0^8 \frac{x^3-2x+1}{\sqrt[3]{x}}dx = \int_0^2 3t(t^9-2t^3+1)dt = \left[\frac{3}{11}t^{11}-\frac{6}{5}x^5+\frac{3}{2}t^2\right]_0^2$

$= \left[\frac{3}{11}x^{\frac{11}{3}}-\frac{6}{5}x^{\frac{5}{3}}+\frac{3}{2}x^{\frac{2}{3}}\right]_0^8 = \frac{28938}{55}$ ■

6. $I=\int_{-3}^3 \frac{2x^2}{2^x+1}dx = \int_{-3}^3 \frac{2x^2}{2^{-x}+1}dx = \int_{-3}^3 \frac{2^x\cdot 2x^2}{2^x+1}dx \ (x \mapsto -x)$

$2I=\int_{-3}^3\left(\frac{2x^2}{2^x+1}+\frac{2^x\cdot 2x^2}{2^x+1}\right)dx = \int_{-3}^3 2x^2dx = \left[\frac{2}{3}x^3\right]_{-3}^3 = 36$

$\therefore I=\int_{-3}^3 \frac{2x^2}{2^x+1}dx = 18$ ■

7. $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{e^{1/x}+1}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{e^{-1/x}+1}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{1/x}\cdot\cos x}{e^{1/x}+1}dx \ (x \mapsto -x)$

$2I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\cos x}{e^{1/x}+1}+\frac{e^{1/x}\cdot\cos x}{e^{1/x}+1}\right)dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x dx = 2$

$\therefore I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{e^{1/x}+1}dx = 1$ ■

8. $\int_0^{\frac{\pi}{2}}\sin 3x\cos 4x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}}(\sin 7x-\sin x)dx = \left[\frac{1}{2}\cos x-\frac{1}{14}\cos 7x\right]_0^{\frac{\pi}{2}} = -\frac{3}{7}$ ■

9. [1] $t^3=\tan x$, $x=\tan^{-1}(t^3)$, $dx=\frac{3t^2}{1+t^6}dt$

[2] $a=t^2$, $da=2tdt$, $a=\tan^{2/3}x$

$$\int_0^{\frac{\pi}{2}} \sqrt[3]{\tan x}\,dx = \int_0^{\infty} \frac{3t^3}{1+t^6}\,dt \quad \cdots \quad [1]$$

$$= \frac{3}{2}\int_0^{\infty} \frac{a}{1+a^3}\,da \quad \cdots \quad [2]$$

$$= \frac{3}{2}\int_0^{\infty} \left(-\frac{1}{3(a+1)} + \frac{a+1}{3(a^2-a+1)}\right)da$$

$$= \frac{1}{2}\int_0^{\infty} \left(\frac{2a-1}{2(a^2-a+1)} + \frac{3}{2(a^2-a+1)} - \frac{1}{a+1}\right)da$$

$$= \left[\frac{1}{4}\ln\frac{a^2-a+1}{(a+1)^2} + \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(a-\frac{1}{2}\right)\right)\right]_0^{\infty}$$

$$= \left[\frac{1}{4}\ln\frac{\tan^{4/3}x - \tan^{2/3}x + 1}{\left(\tan^{2/3}x+1\right)^2} + \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(\tan^{2/3}x - \frac{1}{2}\right)\right)\right]_0^{\frac{\pi}{2}}$$

$$= \lim_{x\to\frac{\pi}{2}}\left\{\frac{1}{4}\ln\frac{\tan^{4/3}x - \tan^{2/3}x + 1}{\left(\tan^{2/3}x+1\right)^2} + \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(\tan^{2/3}x - \frac{1}{2}\right)\right)\right\} - \frac{\sqrt{3}}{2}\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$= \lim_{t\to\infty}\left\{\frac{1}{4}\ln\frac{t^{4/3} - t^{2/3} + 1}{\left(t^{2/3}+1\right)^2} + \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(t^{2/3} - \frac{1}{2}\right)\right)\right\} + \frac{\sqrt{3}}{12}\pi$$

$$= \frac{1}{4}\ln 1 + \frac{\sqrt{3}}{4}\pi + \frac{\sqrt{3}}{12}\pi = \frac{\pi}{\sqrt{3}} \quad ■$$

10. [1] $t^3=\tan x$, $x=\tan^{-1}(t^3)$, $dx=\dfrac{3t^2}{1+t^6}dt$

[2] $a=t^2$, $da=2tdt$, $a=\tan^{2/3}x$

$$\int_0^{\frac{\pi}{4}} \sqrt[3]{\tan x}\,dx = \int_0^{1} \frac{3t^3}{1+t^6}\,dt \quad \cdots \quad [1]$$

$$= \frac{3}{2}\int_0^{1} \frac{a}{1+a^3}\,da \quad \cdots \quad [2]$$

$$=\frac{3}{2}\int_0^1\left(-\frac{1}{3(a+1)}+\frac{a+1}{3(a^2-a+1)}\right)da$$

$$=\frac{1}{2}\int_0^1\left(\frac{2a-1}{2(a^2-a+1)}+\frac{3}{2(a^2-a+1)}-\frac{1}{a+1}\right)da$$

$$=\left[\frac{1}{4}\ln\frac{a^2-a+1}{(a+1)^2}+\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(a-\frac{1}{2}\right)\right)\right]_0^1$$

$$=\left[\frac{1}{4}\ln\frac{\tan^{4/3}x-\tan^{2/3}x+1}{(\tan^{2/3}x+1)^2}+\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(\tan^{2/3}x-\frac{1}{2}\right)\right)\right]_0^{\frac{\pi}{4}}$$

$$=\left(-\frac{1}{2}\ln 2+\frac{\sqrt{3}}{12}\pi\right)-\left(0-\frac{\sqrt{3}}{12}\pi\right)=\frac{\sqrt{3}}{6}\pi-\frac{1}{2}\ln 2 \ \blacksquare$$

11. $$\int_0^1\frac{1}{x^x}dx=\int_0^1 e^{-x\ln x}dx=\int_0^1\sum_{n=0}^{\infty}\frac{(-x\ln x)^n}{n!}dx=\sum_{n=0}^{\infty}\int_0^1\frac{(-x\ln x)^n}{n!}dx$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^1 x^n(\ln x)^n dx$$

한편

$$\int_0^1 x^n(\ln x)^n dx=\left[\frac{1}{n+1}x^{n+1}(\ln x)^n\right]_0^1-\int_0^1\frac{1}{n+1}x^{n+1}\cdot\frac{n(\ln x)^{n-1}}{x}dx$$

$$=-\frac{n}{n+1}\int_0^1 x^n(\ln x)^{n-1}dx=(-1)^2\frac{n(n-1)}{(n+1)^2}\int_0^1 x^n(\ln x)^{n-2}dx$$

$$=(-1)^3\frac{n(n-1)(n-2)}{(n+1)^3}\int_0^1 x^n(\ln x)^{n-3}dx=\cdots$$

$$=(-1)^n\frac{n!}{(n+1)^n}\int_0^1 x^n(\ln x)^{n-n}dx=(-1)^n\frac{n!}{(n+1)^{n+1}}$$

이므로

$$\therefore\int_0^1\frac{1}{x^x}dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^1 x^n(\ln x)^n dx=\sum_{n=0}^{\infty}(n+1)^{-(n+1)}=\sum_{n=1}^{\infty}\frac{1}{n^n} \ \blacksquare$$

12. $I=\int_0^1 \frac{1}{1+\left(1-\frac{1}{x}\right)^{2022}}dx=\int_0^1 \frac{1}{1+\left(\frac{x-1}{x}\right)^{2022}}dx=\int_0^1 \frac{x^{2022}}{x^{2022}+(x-1)^{2022}}dx$

$=\int_0^1 \frac{(1-x)^{2022}}{x^{2022}+(x-1)^{2022}}dx \ \ (x \mapsto 1-x)$

$=\frac{1}{2}\int_0^1\left(\frac{x^{2022}}{x^{2022}+(1-x)^{2022}}+\frac{(1-x)^{2022}}{x^{2022}+(1-x)^{2022}}\right)dx=\frac{1}{2}$ ■

13. $I=\int_{-\infty}^{\infty} e^{-x^2}dx=2\int_0^{\infty} e^{-x^2}dx$ (Gaussian Integral, 가우스 적분)

$I^2=4\int_0^{\infty} e^{-x^2}dx \cdot \int_0^{\infty} e^{-y^2}dy=4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy$

$=4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}rdrd\theta \ \ \left(r=x^2+y^2,\ \theta=\tan^{-1}\left(\frac{y}{x}\right),\ dxdy=rdrd\theta\right)$

$=4\int_0^{\frac{\pi}{2}}d\theta \cdot \int_0^{\infty} re^{-r^2}dr=2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty}=\pi$

$\therefore I=\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$ ■

14. $z=\frac{x-\mu}{\sigma},\ dz=\frac{dx}{\sigma}$ ($N(\mu,\sigma^2)$을 따르는 정규분포의 확률밀도함수)

$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{z^2}{2}}dz$

$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-y^2}dy \ \ (z=\sqrt{2}y,\ dz=\sqrt{2}dy)$

$=1$ ■

15. $\int_0^{\infty}4\pi r^2|\psi(r)|^2dr=\frac{4}{a_0^3}\int_0^{\infty}r^2e^{-\frac{2r}{a_0}}dr=\frac{4}{a_0^3}\left\{\left[-\frac{a_0}{2}r^2e^{-\frac{2r}{a_0}}\right]_0^{\infty}+\frac{a_0}{2}\int_0^{\infty}2re^{-\frac{2r}{a_0}}dr\right\}$

$$= \frac{4}{a_0^3}\left\{0 + a_0\left[-\frac{a_0}{2} r e^{-\frac{2r}{a_0}}\right]_0^\infty + \frac{a_0^2}{2}\int_0^\infty e^{-\frac{2r}{a_0}} dr\right\}$$

$$= \frac{4}{a_0^3}\left\{\frac{a_0^2}{2}\left[-\frac{a_0}{2} e^{-\frac{2r}{a_0}}\right]_0^\infty\right\} = \frac{4}{a_0^3}\left(\frac{a_0^3}{4}\right) = 1 \ \blacksquare$$

$\psi(r)$은 거리 r에 따른 수소원자의 바닥상태 파동함수이다. 위 적분은 수소원자의 전구간에서 전자가 발견될 확률이 1임을 의미한다.

16. $$\int_{\frac{a_0}{2}}^\infty 4\pi r^2 |\psi(r)|^2 dr = \frac{4}{a_0^3}\int_{\frac{a_0}{2}}^\infty r^2 e^{-\frac{2r}{a_0}} dr = \frac{4}{a_0^3}\left\{\left[-\frac{a_0}{2} r^2 e^{-\frac{2r}{a_0}}\right]_{\frac{a_0}{2}}^\infty + \frac{a_0}{2}\int_{\frac{a_0}{2}}^\infty 2r e^{-\frac{2r}{a_0}} dr\right\}$$

$$= \frac{4}{a_0^3}\left\{\frac{a_0^3}{8e} + a_0\left[-\frac{a_0}{2} r e^{-\frac{2r}{a_0}}\right]_{\frac{a_0}{2}}^\infty + \frac{a_0^2}{2}\int_{\frac{a_0}{2}}^\infty e^{-\frac{2r}{a_0}} dr\right\}$$

$$= \frac{4}{a_0^3}\left\{\frac{3a_0^3}{8e} + \frac{a_0^2}{2}\left[-\frac{a_0}{2} e^{-\frac{2r}{a_0}}\right]_{\frac{a_0}{2}}^\infty\right\} = \frac{4}{a_0^3}\left(\frac{5a_0^3}{8e}\right) = \frac{5}{2e} \simeq 91.97\% \ \blacksquare$$

$\psi(r)$은 거리 r에 따른 수소원자의 바닥상태 파동함수이다. 위 적분은 수소원자에서 전자가 핵으로부터 $\frac{a_0}{2}$ 보다 멀리서 발견될 확률이 약 91.97%임을 의미한다. (a_0는 보어 반지름이다.) [2021 물리인증 1급 기출]

17. $$W_n := \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot \sin x dx$$

$$= [-\cos x \cdot \sin^{n-1} x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x \cos x \cdot (-\cos x) dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1 - \sin^2 x) dx$$

$$= (n-1)(W_{n-2} - W_n)$$

$$n W_n = (n-1) W_{n-2}, \quad W_n = \frac{n-1}{n} W_{n-2} \ (n \geq 2)$$

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$$W_0=\int_0^{\frac{\pi}{2}} 1dx=\frac{\pi}{2}, \quad W_1=\int_0^{\frac{\pi}{2}} \sin x dx=1$$

이므로

$$W_n=\frac{n-1}{n}W_{n-2}=\frac{(n-1)(n-3)}{n(n-2)}W_{n-4}= \cdots$$

$$=\begin{cases} \dfrac{(n-1)(n-3)\cdots 1}{n(n-2)\cdots 2}\cdot W_0 & (n=2m) \\ \dfrac{(n-1)(n-3)\cdots 2}{n(n-2)\cdots 1}\cdot W_1 & (n=2m+1) \end{cases}$$

$$\therefore W_n=\frac{(n-1)!!}{n!!}\cdot\left(\frac{\pi}{2}\right)^{\frac{1+(-1)^n}{2}} \ (n>0),\ W_0=\frac{\pi}{2} \ \text{(Wallis' Formula)}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{10}x dx = W_{10}=\frac{9\cdot 7\cdot 5\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2}=\frac{63\pi}{512} \ \blacksquare$$

18. $I(a):=\int_0^1 \frac{x^a-1}{\ln x}dx$, 파인만 적분 테크닉을 적용하면

$$\frac{d}{da}I(a)=\int_0^1 \frac{\partial}{\partial a}\left(\frac{x^a-1}{\ln x}\right)dx=\int_0^1 x^a dx=\left[\frac{1}{a+1}x^{a+1}\right]_0^1=\frac{1}{a+1}$$

$$I(a)=\int \frac{1}{a+1}da=\ln|a+1|+C$$

이때 $I(0)=\int_0^1 0\,dx=0$이므로 $C=0$이다.

$$\therefore I(2022)=\int_0^1 \frac{x^{2022}-1}{\ln x}dx=\ln 2023 \ \blacksquare$$

19. $(1+x)^{-1}$의 이항급수를 구하고 양변을 적분하면

$$(1+x)^{-1}=\sum_{n=0}^{\infty}\binom{-1}{n}\cdot x^n=\sum_{n=0}^{\infty}\prod_{i=0}^{n-1}(-1-i)\cdot\frac{x^n}{n!}=\sum_{n=0}^{\infty}(-1)^n\cdot x^n \quad (|x|<1)$$

$$\int (1+x)^{-1}dx = \ln|1+x| = \int \sum_{n=0}^{\infty}(-1)^n x^n dx + C = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} + C$$

$x=0$을 대입하면 $0=0+C$에서 $C=0$이다.

$$\therefore \int_0^1 \frac{\ln x}{x-1}dx = \int_{-1}^0 \frac{\ln(t+1)}{t}dt = \int_{-1}^0 \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}t^n dt = \left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}t^{n+1}\right]_{-1}^0$$

$$= \left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}0^{n+1}\right] - \left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}(-1)^{n+1}\right] = \sum_{n=0}^{\infty}\frac{1}{(n+1)^2}$$

$= \zeta(2) = \frac{\pi^2}{6}$ ■ ($\because$ 바젤 문제에 의해 $\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$)

20. $I(\alpha) := \int_0^{\infty}\frac{\sin x}{x}e^{-\alpha x}dx$, $\frac{d}{d\alpha}I(\alpha) = \int_0^{\infty}\frac{\partial}{\partial \alpha}\frac{\sin x}{x}e^{-\alpha x}dx = -\int_0^{\infty}e^{-\alpha x}\sin x dx$

$$= -[-e^{-\alpha x}\cos x]_0^{\infty} + \int_0^{\infty}\alpha e^{-\alpha x}\cos x dx = -1 + [\alpha e^{-\alpha x}\sin x]_0^{\infty} + \int_0^{\infty}\alpha^2 e^{-\alpha x}\sin x dx$$

$$= -1 - \alpha^2 I'(\alpha), \quad I'(\alpha) = -\frac{1}{1+\alpha^2}$$

$I(\alpha) = \int I'(\alpha)d\alpha = C - \tan^{-1}\alpha \ (C \in \mathbb{R})$, $0 = \lim_{\alpha \to \infty} I(\alpha) = C - \frac{\pi}{2}$에서 $C = \frac{\pi}{2}$이다.

$\therefore \int_{-\infty}^{\infty}\frac{\sin x}{x}dx = 2\int_0^{\infty}\frac{\sin x}{x}dx = 2I(0) = \pi$ ■

21. $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx = \left[-\frac{\sin^2 x}{x}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{2\sin x\cos x}{x}dx = \int_{-\infty}^{\infty}\frac{\sin 2x}{x}dx$

$$= \int_{-\infty}^{\infty}\frac{\sin t}{t}dt \quad (t = 2x, \ dt = 2dx)$$

$$I(\alpha) := \int_0^{\infty}\frac{\sin x}{x}e^{-\alpha x}dx, \ \frac{d}{d\alpha}I(\alpha) = \int_0^{\infty}\frac{\partial}{\partial \alpha}\frac{\sin x}{x}e^{-\alpha x}dx = -\int_0^{\infty}e^{-\alpha x}\sin x dx$$

$$= -[-e^{-\alpha x}\cos x]_0^{\infty} + \int_0^{\infty}\alpha e^{-\alpha x}\cos x dx = -1 + [\alpha e^{-\alpha x}\sin x]_0^{\infty} + \int_0^{\infty}\alpha^2 e^{-\alpha x}\sin x dx$$

$$= -1-\alpha^2 I'(\alpha), \quad I'(\alpha) = -\frac{1}{1+\alpha^2}$$

$$I(\alpha) = \int I'(\alpha)d\alpha = C - \tan^{-1}\alpha \ \ (C \in \mathbb{R}), \quad 0 = \lim_{\alpha\to\infty} I(\alpha) = C - \frac{\pi}{2}\text{에서 } \ C = \frac{\pi}{2}\text{이다.}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}dx = \int_{-\infty}^{\infty} \frac{\sin x}{x}dx = 2\int_0^{\infty} \frac{\sin x}{x}dx = 2I(0) = \pi \ \blacksquare$$

22. $I = \int_0^1 \sqrt[3]{2x^3-3x^2-x+1}\,dx = \int_0^1 \sqrt[3]{2(1-x)^3-3(1-x)^2-(1-x)+1}\,dx \ \ (x \mapsto 1-x)$

$$= \int_0^1 -\sqrt[3]{2x^3-3x^2-x+1}\,dx = -I$$

$$\therefore I = \int_0^1 \sqrt[3]{2x^3-3x^2-x+1}\,dx = 0 \ \blacksquare$$

23. $x = \tan t, \ dx = \sec^2 t dt, \ \dfrac{dx}{1+x^2} = dt$

$$I = \int_0^1 \frac{\ln(x+1)}{x^2+1}dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan t)dt = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-t\right)\right)dt \ \ \left(t \mapsto \frac{\pi}{4}-t\right)$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan t}{1+\tan t}\right)dt = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan t}\right)dt = \int_0^{\frac{\pi}{4}} \{\ln 2 - \ln(1+\tan t)\}dt$$

$$2I = \int_0^{\frac{\pi}{4}} \{\ln(1+\tan t) + (\ln 2 - \ln(1+\tan t))\}dt = \int_0^{\frac{\pi}{4}} \ln 2 dt = \frac{\pi}{4}\ln 2$$

$$\therefore I = \int_0^1 \frac{\ln(x+1)}{x^2+1}dx = \frac{\pi}{8}\ln 2 \ \blacksquare$$

24. $I = \int_0^2 \dfrac{\ln(x+1)}{x^2-x+1}dx = \int_1^3 \dfrac{\ln t}{t^2-3t+3}dt \ \ (t = x+1, \ dt = dx)$

$$= \int_3^1 \frac{\ln(3u^{-1})}{9u^{-2}-9u^{-1}+3}\left(-\frac{3}{u^2}du\right) \ \ \left(u = \frac{3}{t}, \ du = -\frac{3}{t^2}dt, \ dt = -\frac{3}{u^2}du\right)$$

$$=\int_1^3 \frac{\ln 3-\ln u}{u^2-3u+3}du=\int_1^3 \frac{\ln 3}{u^2-3u+3}du-\int_1^3 \frac{\ln u}{u^2-3u+3}du=\int_1^3 \frac{\ln 3}{u^2-3u+3}du-I$$

$$=\frac{\ln 3}{2}\int_1^3 \frac{1}{u^2-3u+3}du=\frac{\ln 3}{2}\int_1^3 \frac{1}{\left(u-\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}du=\frac{\ln 3}{2}\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-3}{\sqrt{3}}\right)\right]_1^3$$

$$=\frac{\ln 3}{\sqrt{3}}\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\frac{\pi\ln 3}{2\sqrt{3}}\ \blacksquare$$

25. $I(\alpha):=\int_0^1 \frac{\ln(\alpha x^2+1)}{x+1}dx$, $\frac{d}{d\alpha}I(\alpha)=\int_0^1 \frac{\partial}{\partial\alpha}\frac{\ln(\alpha x^2+1)}{x+1}dx=\int_0^1 \frac{x^2}{(\alpha x^2+1)(x+1)}dx$

$$=\int_0^1\left(\frac{x-1}{(\alpha+1)(\alpha x^2+1)}+\frac{1}{(\alpha+1)(x+1)}\right)dx=\frac{1}{\alpha+1}\int_0^1\left(\frac{x}{\alpha x^2+1}-\frac{1}{\alpha x^2+1}+\frac{1}{x+1}\right)dx$$

$$=\frac{1}{\alpha+1}\left[\frac{1}{2\alpha}\ln|\alpha x^2+1|-\frac{1}{\sqrt{a}}\tan^{-1}(\sqrt{\alpha}\,x)+\ln|x+1|\right]_0^1$$

$$=\frac{1}{\alpha+1}\left(\frac{\ln(\alpha+1)}{2\alpha}-\frac{\tan^{-1}(\sqrt{\alpha}\,x)}{\sqrt{\alpha}}+\ln 2\right)$$

$$\therefore \int_0^1 \frac{\ln(x^2+1)}{x+1}dx=I(1)=\frac{1}{2}\int_0^1 \frac{\ln(\alpha+1)}{\alpha(\alpha+1)}d\alpha-\int_0^1 \frac{\tan^{-1}(\alpha)}{\sqrt{\alpha}(\alpha+1)}d\alpha+\int_0^1 \frac{\ln 2}{\alpha+1}d\alpha$$

$$=\frac{1}{2}\left(\int_0^1 \frac{\ln(\alpha+1)}{\alpha}d\alpha-\int_0^1 \frac{\ln(\alpha+1)}{\alpha+1}d\alpha\right)-2\left[\frac{1}{2}\left(\tan^{-1}(\alpha)\right)^2\right]_0^1+\ln 2\left[\ln(\alpha+1)\right]_0^1$$

$$=\frac{1}{2}\left(\int_0^1\sum_{n=0}^{\infty}\frac{(-1)^n\alpha^n}{n+1}d\alpha-\left[\frac{1}{2}(\ln(\alpha+1))^2\right]_0^1\right)-\frac{\pi^2}{16}+(\ln 2)^2$$

$$=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\int_0^1\alpha^n d\alpha-\frac{1}{2}(\ln 2)^2\right)-\frac{\pi^2}{16}+(\ln 2)^2$$

$$=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}\right)-\frac{\pi^2}{16}+\frac{3}{4}(\ln 2)^2=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{2n^2}\right)-\frac{\pi^2}{16}+\frac{3}{4}(\ln 2)^2$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{\pi^2}{16}+\frac{3}{4}(\ln 2)^2=\frac{1}{4}\zeta(2)-\frac{\pi^2}{16}+\frac{3}{4}(\ln 2)^2$$

$= \frac{3}{4}(\ln 2)^2 - \frac{\pi^2}{48}$ ■ ($\because$ 바젤 문제에 의해 $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$)

26. $I = \int_0^{\frac{\pi}{2}} \ln(\cos x)dx = \int_0^{\frac{\pi}{2}} \ln(\sin x)dx$ ($x \mapsto \frac{\pi}{2} - x$)

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin x \cos x)dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin 2x\right)dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin 2x)dx - \frac{\pi}{4}\ln 2$

$= \frac{1}{4}\int_0^{\pi} \ln(\sin t)dt - \frac{\pi}{4}\ln 2$ ($t = 2x$, $dt = 2dx$)

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin t)dt - \frac{\pi}{4}\ln 2 = \frac{1}{2}I - \frac{\pi}{4}\ln 2$, $\quad \frac{1}{2}I = -\frac{\pi}{4}\ln 2$

$\therefore I = \int_0^{\frac{\pi}{2}} \ln(\cos x)dx = -\frac{\pi}{2}\ln 2$ ■

27. sol 1) $I = \int_0^1 \frac{\ln x}{x+1}dx = \int_0^1 \frac{\ln x}{1-x}dx - \int_0^1 \frac{2x\ln x}{1-x^2}dx$

$= \int_0^1 \frac{\ln x}{1-x}dx - \frac{1}{2}\int_0^1 \frac{\ln t}{1-t}dt$ ($t = x^2$, $dt = 2xdx$)

$= \frac{1}{2}\int_0^1 \frac{\ln t}{1-t}dt$

$(1+x)^{-1}$의 이항급수를 구하고 양변을 적분하면

$(1+x)^{-1} = \sum_{n=0}^{\infty}\binom{-1}{n} \cdot x^n = \sum_{n=0}^{\infty}\prod_{i=0}^{n-1}(-1-i) \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty}(-1)^n \cdot x^n \quad (|x| < 1)$

$\int (1+x)^{-1}dx = \ln|1+x| = \int \sum_{n=0}^{\infty}(-1)^n x^n dx + C = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} + C$

$x = 0$을 대입하면 $0 = 0 + C$에서 $C = 0$이다.

$\int_0^1 \frac{\ln x}{x-1}dx = \int_{-1}^0 \frac{\ln(t+1)}{t}dt = \int_{-1}^0 \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}t^n dt = \left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}t^{n+1}\right]_{-1}^0$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} 0^{n+1} \right] - \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} (-1)^{n+1} \right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

$= \zeta(2) = \frac{\pi^2}{6}$ ($\because$ 바젤 문제에 의해 $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$)

$$\therefore I = \frac{1}{2} \int_0^1 \frac{\ln t}{1-t} dt = -\frac{\pi^2}{12}$$

sol 2) $\int_0^1 x^a dx = \frac{1}{a+1}$의 양변을 a로 미분하면

$$\frac{d}{da} \int_0^1 x^a dx = \int_0^1 \frac{\partial}{\partial a} x^a dx = \int_0^1 x^a \ln x dx = -\frac{1}{(a+1)^2}$$

한편 $|x| < 1$일 때 $\sum_{a=0}^{\infty} (-1)^a x^a = \frac{1}{1+x}$ 임을 이용하면

$$\sum_{a=0}^{\infty} (-1)^a \int_0^1 x^a \ln x dx = \int_0^1 \sum_{a=0}^{\infty} (-1)^a x^a \ln x dx = -\sum_{a=0}^{\infty} \frac{(-1)^a}{(a+1)^2}$$

$$\therefore \int_0^1 \frac{\ln x}{x+1} dx = -\sum_{a=0}^{\infty} \frac{(-1)^a}{(a+1)^2} = \sum_{a=1}^{\infty} \frac{(-1)^a}{a^2} = \sum_{a=1}^{\infty} \frac{1}{2a^2} - \sum_{a=1}^{\infty} \frac{1}{a^2}$$

$$= -\frac{1}{2} \sum_{a=1}^{\infty} \frac{1}{a^2} = -\frac{1}{2} \zeta(2) = -\frac{\pi^2}{12} \ \blacksquare$$

28. $n \in \mathbb{N} \cup \{0\}$, $a_n := \int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx$

$$a_n = \int_0^{\frac{\pi}{2}} \cos^n x \{\cos(n+1)x \cos x + \sin(n+1)x \sin x\} dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n+1} x \cos(n+1)x dx + \int_0^{\frac{\pi}{2}} \cos^n x \sin(n+1)x \sin x dx$$

$$= a_{n+1} + \left[\sin(n+1)x \cdot \frac{-1}{n+1} \cos^{n+1} x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos(n+1)x (-\cos^{n+1} x) dx$$

$$= 2a_{n+1}$$

$$a_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2},\ \ a_1 = \int_0^{\frac{\pi}{2}} \cos^2 dx = \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$a_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx = \pi \cdot \left(\frac{1}{2}\right)^{n+1}$$

$$\therefore a_{2022} = \int_0^{\frac{\pi}{2}} \cos^{2022} x \cos 2022x dx = \frac{\pi}{2^{2023}} \ ■$$

29. sol 1) $t = -\ln x,\ x = e^{-t},\ dx = -e^{-t}dt$

$$\int_0^1 (\ln x)^{2022} dx = \int_0^{\infty} t^{2022} e^{-t} dt = \Gamma(2023) = 2022! \ ■$$

sol 2) $J_n := \int_0^1 (-\ln x)^n dx = [x \cdot (-\ln x)^n]_0^1 - \int_0^1 x \cdot n(-\ln x)^{n-1} \cdot \frac{-1}{x} dx$

$$= 0 - \lim_{x \to 0}\{x \cdot (-\ln x)^n\} + n\int_0^1 (-\ln x)^{n-1} dx = nJ(n-1)$$

$(\because x = e^{-t},\ \lim_{x \to 0}\{x \cdot (-\ln x)^n\} = \lim_{t \to \infty} \frac{t^n}{e^t} = 0$ by l'Hospital's rule)

$$J_n = nJ_{n-1} = n(n-1)J_{n-2} = \ \cdots \ = n(n-1)\cdots 1 \cdot J_0 = n!$$

$$\therefore \int_0^1 (\ln x)^{2022} dx = J_{2022} = 2022! \ ■$$

30. $\int_1^e \frac{\ln x}{\sqrt{x}} dx = [\ln x \cdot 2\sqrt{x}]_1^e - \int_1^e \frac{2\sqrt{x}}{x} dx = 2\sqrt{e} - [4\sqrt{x}]_1^e = 4 - 2\sqrt{e}$ ■

31. $I(\alpha) := \int_0^{\frac{\pi}{2}} \ln(a + \tan^2 x) dx,\ \frac{d}{d\alpha} I(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \alpha} \ln(a + \tan^2 x) dx = \int_0^{\frac{\pi}{2}} \frac{1}{a + \tan^2 x} dx$

$$= \int_0^{\infty} \frac{1}{(t^2+1)(a+t^2)} dt = \frac{1}{a-1} \int_0^{\infty} \left(\frac{1}{t^2+1} - \frac{1}{t^2+a}\right) dt \ \ (t = \tan x,\ dt = \sec^2 x dx)$$

$$= \frac{1}{a-1}\left[\tan^{-1}t - \frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{t}{\sqrt{a}}\right)\right]_0^\infty = \frac{\pi}{2}\left(\frac{1}{a-1} - \frac{1}{\sqrt{a}(a-1)}\right)$$

$$I(\alpha) = \int \frac{\pi}{2}\left(\frac{1}{a-1} - \frac{1}{\sqrt{a}(a-1)}\right)da = \frac{\pi}{2}\int\left(\frac{1}{a-1} - \frac{1}{\sqrt{a}(a-1)}\right)dx$$

$$= \frac{\pi}{2}\left(\ln(a-1) + \ln\left(\frac{\sqrt{a}+1}{\sqrt{a}-1}\right)\right) = \pi\ln(\sqrt{a}+1)$$

$$(\because I(0) = 2\int_0^{\frac{\pi}{2}} \ln(\tan x)dx = 2\int_0^{\frac{\pi}{2}} \ln(\cot x)dx = -I(0) = 0)$$

$$\therefore \int_0^{\frac{\pi}{2}} \ln(2+\tan^2 x)dx = \pi\ln(\sqrt{2}+1) \ \blacksquare$$

32. $I(\alpha) := \int_0^\infty \frac{\alpha x - \sin(\alpha x)}{x^3(x^2+1)}dx$, $\frac{d^3}{d\alpha^3}I(\alpha) = \int_0^\infty \frac{\partial^3}{\partial\alpha^3}\frac{\alpha x - \sin(\alpha x)}{x^3(x^2+1)}dx$

$$= \int_0^\infty \frac{\partial^2}{\partial\alpha^2}\frac{1-\cos(\alpha x)}{x^2(x^2+1)}dx = \int_0^\infty \frac{\partial}{\partial\alpha}\frac{\sin(\alpha x)}{x(x^2+1)}dx = \int_0^\infty \frac{\cos(\alpha x)}{x^2+1}dx$$

$$J(\alpha) := \int_{-\infty}^\infty \frac{\cos\alpha x}{x^2+1}dx = \left[\frac{\sin(\alpha x)}{\alpha(x^2+1)}\right]_{-\infty}^\infty + \frac{2}{\alpha}\int_{-\infty}^\infty \frac{x\sin(\alpha x)}{(x^2+1)^2}dx$$

$$= \frac{2}{\alpha}\int_{-\infty}^\infty \frac{x\sin(\alpha x)}{(x^2+1)^2}dx, \quad \alpha J(\alpha) = 2\int_{-\infty}^\infty \frac{x\sin(\alpha x)}{(x^2+1)^2}dx$$

양변을 미분하면

$$\frac{d}{d\alpha}(\alpha J(\alpha)) = J(\alpha) + \alpha J'(\alpha) = 2\int_{-\infty}^\infty \frac{\partial}{\partial\alpha}\frac{x\sin(\alpha x)}{(x^2+1)^2}dx = 2\int_{-\infty}^\infty \frac{x^2\cos(\alpha x)}{(x^2+1)^2}dx$$

$$= 2\int_{-\infty}^\infty \frac{\cos(\alpha x)}{x^2+1}dx - 2\int_{-\infty}^\infty \frac{\cos(\alpha x)}{(x^2+1)^2}dx = 2J(\alpha) - 2\int_{-\infty}^\infty \frac{\cos(\alpha x)}{(x^2+1)^2}dx$$

$\alpha J'(\alpha) - J(\alpha) = -2\int_{-\infty}^\infty \frac{\cos(\alpha x)}{(x^2+1)^2}dx$, 다시 양변을 α로 미분하면

$$\alpha J''(\alpha) = -2\int_{-\infty}^\infty \frac{\partial}{\partial\alpha}\frac{\cos(\alpha x)}{(x^2+1)^2}dx = 2\int_{-\infty}^\infty \frac{x\sin(\alpha x)}{(x^2+1)^2}dx = \alpha J(\alpha)$$

$J''(\alpha) = J(\alpha)$라는 미분방정식을 얻는다.

$J(\alpha)$의 이계도함수가 $J(\alpha)$와 동일해야 하므로 $J(\alpha)$는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4에 대하여
$J(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$와 $J(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i \sin x$를 적용하면 두 경우가 동일함을 알 수 있다.)

$$J(\alpha) = C_1 e^{\alpha} + C_2 e^{-\alpha} \quad (\alpha > 0)$$

한편 $J(0) = C_1 + C_2 = \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \left[\tan^{-1} x\right]_{-\infty}^{\infty} = \pi$,

$J(\alpha) = \frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{x \sin(\alpha x)}{(x^2+1)^2} dx$에서 $\lim_{\alpha \to \infty} J(\alpha) = 0$, $C_1 = 0$이고 $C_2 = \pi$이다.

$\therefore J(\alpha) = \frac{\pi}{e^{\alpha}}$

$\frac{d^3}{d\alpha^3} I(\alpha) = \int_0^{\infty} \frac{\cos(\alpha x)}{x^2+1} dx = \frac{1}{2} J(\alpha) = \frac{\pi}{2e^{\alpha}}$이고

$I(0) = I'(0) = I''(0) = 0$이므로

$I''(\alpha) = \frac{\pi}{2}(1 - e^{-\alpha})$, $\quad I'(\alpha) = \frac{\pi}{2}(\alpha + e^{-\alpha} - 1)$, $\quad I(\alpha) = \frac{\pi}{2}\left(\frac{1}{2}\alpha^2 - e^{-\alpha} - \alpha + 1\right)$이다.

한편 $x = 2t$, $dx = 2dt$의 치환을 하면

$$\int_0^{\infty} \frac{x - \sin x}{x^3(x^2+4)} dx = \int_0^{\infty} \frac{2t - \sin(2t)}{8t^3(4t^2+4)} \cdot 2dt = \frac{1}{16} \int_0^{\infty} \frac{2t - \sin(2t)}{t^3(t^2+1)} dt = \frac{1}{16} I(2)$$

$= \frac{\pi}{32}(1 - e^{-2})$ ■

33. $u = \tan \frac{x}{2}$, $du = \frac{1}{2} \sec^2 \frac{x}{2} dx$

$$\int_0^{\pi}\frac{1-\sin x}{1+\sin x}dx=\int_0^{\infty}\frac{2\left(1-\frac{2u}{u^2+1}\right)}{(u^2+1)\left(\frac{2u}{u^2+1}+1\right)}du=\int_0^{\infty}\frac{2(u-1)^2}{(u^2+1)(u+1)^2}du$$

$$=2\int_0^{\infty}\left(\frac{2}{(u+1)^2}-\frac{1}{u^2+1}\right)du=2\left[-\frac{2}{u+1}-\tan^{-1}u\right]_0^{\infty}=4-\pi \ ■$$

34. $$\int_0^{\frac{3}{5}}\frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx=\int_0^{\frac{3}{5}}\left(\frac{e^x}{(1-x)\sqrt{1-x^2}}+e^x\sqrt{\frac{1+x}{1-x}}\right)dx$$

$$=\int_0^{\frac{3}{5}}\left(e^x\left(\sqrt{\frac{1+x}{1-x}}\right)'+(e^x)'\sqrt{\frac{1+x}{1-x}}\right)dx=\left[e^x\sqrt{\frac{1+x}{1-x}}\right]_0^{\frac{3}{5}}=2e^{\frac{3}{5}}-1 \ ■$$

35. $$\int_0^{\frac{\pi}{2}}\sin 2022x\cdot\sin^{2020}x\,dx=\int_0^{\frac{\pi}{2}}\sin(2021x+x)\cdot\sin^{2020}x\,dx$$

$$=\int_0^{\frac{\pi}{2}}\sin 2021x\cdot\cos x\cdot\sin^{2020}x\,dx+\int_0^{\frac{\pi}{2}}\cos 2021x\cdot\sin^{2021}x\,dx$$

$$=\sin 2021x\cdot\frac{1}{2021}\sin^{2021}x-\frac{2021}{2021}\int_0^{\frac{\pi}{2}}\cos 2021x\cdot\sin^{2021}x\,dx+\int_0^{\frac{\pi}{2}}\cos 2021x\cdot\sin^{2021}x\,dx$$

$$=\left[\frac{1}{2021}\sin 2021x\cdot\sin^{2021}x\,dx\right]_0^{\frac{\pi}{2}}=\frac{1}{2021} \ ■$$

36. $t=x^2+\frac{1}{x^2}-1$, $dt=2\left(x-\frac{1}{x^3}\right)dx$

$$\int_1^{\sqrt{2}}\frac{x^4-1}{x^2\sqrt{x^4-x^2+1}}dx=\int_1^{\sqrt{2}}\frac{x^4-1}{x^3\sqrt{x^2+\frac{1}{x^2}-1}}dx=\int_0^{\sqrt{2}}\frac{x-1/x^3}{\sqrt{x^2+\frac{1}{x^2}-1}}dx$$

$$=\frac{1}{2}\int_1^{\frac{3}{2}}\frac{1}{\sqrt{t}}dt=\left[\sqrt{t}\right]_1^{\frac{3}{2}}=\frac{\sqrt{6}}{2}-1 \ ■$$

37. $u=\cos x$, $du=-\sin x\,dx$

$$\int_0^{\frac{\pi}{3}} \frac{\sin 2x}{2+\cos x} dx = (-2)\int_0^{\frac{\pi}{3}} \frac{\cos x(-\sin x)}{2+\cos x} dx = (-2)\int_1^{\frac{1}{2}} \frac{u}{2+u} du = 2\int_{\frac{1}{2}}^1 \left(1-\frac{2}{u+2}\right) du$$

$$= 2\left[u - 2\ln|u+2|\right]_{\frac{1}{2}}^1 = 2(1-\ln 9) - 2\left(\frac{1}{2} - 2\ln\frac{5}{2}\right) = 1 - \ln\left(\frac{1296}{625}\right) \ \blacksquare$$

38. (19)의 reduction formula를 이용하면

$$\int \tan^n x dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2} x dx \ \text{에서}$$

$$\int_0^{\frac{\pi}{4}} \tan^9 x dx = \left[\frac{\tan^8 x}{8}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^7 x dx = \frac{1}{8} - \left[\frac{\tan^6 x}{6}\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tan^5 x dx$$

$$= \frac{1}{8} - \frac{1}{6} + \left[\frac{\tan^4 x}{4}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^3 x dx = \frac{1}{8} - \frac{1}{6} + \frac{1}{4} - \left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tan x dx$$

$$= \frac{1}{8} - \frac{1}{6} + \frac{1}{4} - \frac{1}{2} + \left[-\ln|\cos x|\right]_0^{\frac{\pi}{4}} = \frac{1}{8} - \frac{1}{6} + \frac{1}{4} - \frac{1}{2} + \ln\sqrt{2} = \frac{1}{2}\ln 2 - \frac{7}{24} \ \blacksquare$$

39. $I = \displaystyle\int_0^1 \frac{x-1}{(1+x^3)\ln x} dx = \frac{1}{2}\int_0^\infty \frac{x-1}{(1+x^3)\ln x} dx \ \left(x \mapsto \frac{1}{x}\right)$

$$J(a) := \int_0^\infty \frac{x^a - 1}{(1+x^3)\ln x} dx, \quad J'(a) = \frac{\partial}{\partial a} J(a) = \int_0^\infty \frac{x^a}{1+x^3} dx = \frac{\pi}{3}\csc\left(\frac{\pi(a+1)}{3}\right)$$

$$\therefore I = \frac{1}{2} J(1) = \frac{1}{2}\int_0^1 J'(a) da = \frac{\pi}{6}\int_0^1 \csc\left(\frac{\pi(a+1)}{3}\right) da = \frac{\ln 3}{2} \ \blacksquare$$

40. sol 1) $u = \dfrac{\sin t}{\sqrt{\cos t}}$, $du = \dfrac{1+\cos^2 t}{2\cos^{3/2} t} dt$

$$\int_0^{\frac{\pi}{2}} \frac{(1+\sec^2 t)\sqrt{\sec t}}{(1+\sec t)^2 - 2} dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos^2 t}{\sqrt{\cos t}\left\{(\cos t+1)^2 - 2\cos^2 t\right\}} dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{1+\cos^2 t}{\sqrt{\cos t}(1+2\cos t - \cos^2 t)} dt = 2\int_0^{\frac{\pi}{2}} \frac{\cos t}{1+2\cos t - \cos^2 t} \cdot \frac{1+\cos^2 t}{2\cos^{3/2} t} dt$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{\frac{1}{\cos t}+2-\cos t} \cdot \frac{1+\cos^2 t}{2\cos^{3/2} t} dt = 2\int_0^{\frac{\pi}{2}} \frac{1}{\frac{1-\cos^2 t}{\cos t}+2} \cdot \frac{1+\cos^2 t}{2\cos^{3/2} t} dt$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{\frac{\sin^2 t}{\cos t}+2} \cdot \frac{1+\cos^2 t}{2\cos^{3/2} t} dt = 2\int_0^{\infty} \frac{1}{u^2+2} du = 2\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\infty} = \frac{\pi}{\sqrt{2}} \ \blacksquare$$

sol 2) $\sec t = x^2$, $\sec t \tan t dt = 2x dx$, $u = \sqrt{x^2 - \frac{1}{x^2}}$, $du = \frac{x^4+1}{x^2\sqrt{x^4-1}}$

$$I = \int_0^{\frac{\pi}{2}} \frac{(1+\sec^2 t)\sqrt{\sec t}}{(1+\sec t)^2 - 2} dt = 2\int_1^{\infty} \frac{x^4+1}{\sqrt{x^4-1}\,(x^4+2x^2-1)} dx = 2\int_0^{\infty} \frac{1}{u^2+2} du$$

$$= 2\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\infty} = \frac{\pi}{\sqrt{2}} \ \blacksquare$$

41. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{2-\sin 2x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{2-2\sin x\cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\cos^2 x - 2\sin x\cos x + \sin^2 x} dx$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+(\cos x - \sin x)^2} dx = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+(\cos x-\sin x)^2} dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+(\cos x - \sin x)^2} dx\right]$$

$$(\because \ x \mapsto \frac{\pi}{2} - x \ : \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+(\cos x - \sin x)^2} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+(\cos x - \sin x)^2} dx)$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{1+(\cos x - \sin x)^2} dx = -\frac{1}{2}\left[\tan^{-1}(\cos x - \sin x)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} \ \blacksquare$$

42. $\displaystyle\int_0^{\infty} (\ln(e^x+1) - x) dx = \int_0^{\infty} \ln\left(\frac{e^x+1}{e^x}\right) dx = \int_0^{\infty} \ln(1+e^{-x}) dx$

$$= \int_0^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^{-nx}}{n} dx = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{(-1)^{n-1} e^{-nx}}{n} dx = \sum_{n=1}^{\infty} \left[\frac{(-1)^n e^{-nx}}{n^2}\right]_0^{\infty}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\zeta(2)}{2} = \frac{\pi^2}{12} \ \blacksquare$$

43. $I=\int_0^\infty \frac{x}{e^x+1}dx=\int_1^\infty \frac{\ln t}{t(1+t)}dt\ \left(x=\ln t,\ dx=\frac{1}{t}dt\right)$

$=-\int_0^1 \frac{\ln u}{1+u}du\ \left(u=\frac{1}{t},\ du=-\frac{1}{u^2}du\right)$

$=\int_0^1 \frac{\ln u}{1-u}du-\int_0^1 \frac{2u\ln u}{1-u^2}du=\int_0^1 \frac{\ln u}{1-u}du-\frac{1}{2}\int_0^1 \frac{\ln y}{1-y}dy\ (y=u^2,\ dy=2udu)$

$=\frac{1}{2}\int_0^1 \frac{\ln u}{1-u}du$

$(1+x)^{-1}$의 이항급수를 구하고 양변을 적분하면

$(1+x)^{-1}=\sum_{n=0}^{\infty}\binom{-1}{n}\cdot x^n=\sum_{n=0}^{\infty}\prod_{i=0}^{n-1}(-1-i)\cdot\frac{x^n}{n!}=\sum_{n=0}^{\infty}(-1)^n\cdot x^n\quad(|x|<1)$

$\int(1+x)^{-1}dx=\ln|1+x|=\int\sum_{n=0}^{\infty}(-1)^n x^n dx+C=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}+C$

$x=0$을 대입하면 $0=0+C$에서 $C=0$이다.

$\int_0^1 \frac{\ln x}{x-1}dx=\int_{-1}^0 \frac{\ln(t+1)}{t}dt=\int_{-1}^0\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}t^n dt=\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}t^{n+1}\right]_{-1}^0$

$=\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}0^{n+1}\right]-\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}(-1)^{n+1}\right]=\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}$

$=\zeta(2)=\frac{\pi^2}{6}$ ($\because$ 바젤 문제에 의해 $\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$)

$I=\int_0^\infty \frac{x}{e^x+1}dx=\int_1^\infty \frac{\ln t}{t(1+t)}dt=\frac{1}{2}\int_0^1 \frac{\ln u}{1-u}du=\frac{\pi^2}{12}$ ■

44. $\int_0^{\frac{\pi}{2}} \frac{\ln(\tan x)}{1-\tan x+\tan^2 x}dx=\int_0^\infty \frac{\ln t}{(1-t+t^2)(1+t^2)}dt\ (t=\tan x,\ dt=\sec^2 x dx)$

$=\int_0^1 \frac{\ln t}{(1-t+t^2)(1+t^2)}dt+\int_1^\infty \frac{\ln t}{(1-t+t^2)(1+t^2)}dt$

$$= \int_0^1 \frac{\ln t}{(1-t+t^2)(1+t^2)}dt - \int_0^1 \frac{u^2 \ln u}{(1-u+u^2)(1+u^2)}du \ \ (u = \frac{1}{t},\ du = -\frac{1}{t^2}dt)$$

$$= \int_0^1 \frac{(1-u^2)\ln u}{(1-u+u^2)(1+u^2)}du = \int_0^1 \frac{2u\ln u}{1+u^2}du - \int_0^1 \frac{(2u-1)\ln u}{1-u+u^2}du$$

$$= [\ln u \ln(u^2+1)]_0^1 - \int_0^1 \frac{1}{u}\ln(u^2+1)du - [\ln u \ln(1-u+u^2)]_0^1 + \int_0^1 \frac{1}{u}\ln(1-u+u^2)du$$

$$= -\int_0^1 \frac{\ln(1+u^2)}{u}du + \int_0^1 \frac{\ln(1-u+u^2)}{u}du$$

$$= -\int_0^1 \frac{\ln(1+u^2)}{u^2}udu + \int_0^1 \frac{\ln(1+u^3)}{u}du - \int_0^1 \frac{\ln(1+u)}{u}du$$

$$= -\frac{1}{2}\int_0^1 \frac{\ln(1+y)}{y}dy + \int_0^1 \frac{\ln(1+u^3)}{u^3}u^2du - \int_0^1 \frac{\ln(1+u)}{u}du \ \ (y = u^2,\ dy = 2udu)$$

$$= -\frac{3}{2}\int_0^1 \frac{\ln(1+u)}{u}du + \frac{1}{3}\int_0^1 \frac{\ln(1+z)}{z}dz \ \ (z = u^3,\ dz = 3u^2du)$$

$$= -\frac{7}{6}\int_0^1 \frac{\ln(1+u)}{u}du = -\frac{7}{6}\int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n u^n}{n+1}du = -\frac{7}{6}\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\int_0^1 u^n du$$

$$= -\frac{7}{6}\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = -\frac{7}{6}\left(\sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\right) = -\frac{7}{12}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$= -\frac{7}{12}\zeta(2) = -\frac{7\pi^2}{72}$ ■ ($\because$ 바젤 문제에 의해 $\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$)

45. $t = x - \frac{1}{x},\ dt = \left(1 + \frac{1}{x^2}\right)dx,\ \frac{dx}{1+x^2} = \frac{x^2}{(1+x^2)^2}dt = \frac{dt}{(x+1/x)^2} = \frac{dt}{t^2+4}$

$$I = \int_{-\infty}^{\infty} \frac{\cos\left(x - \frac{1}{x}\right)}{1+x^2}dx = 2\int_0^{\infty} \frac{\cos\left(x - \frac{1}{x}\right)}{1+x^2}dx = 2\int_{-\infty}^{\infty} \frac{\cos t}{t^2+4}dt$$

$$J(\alpha) := \int_{-\infty}^{\infty} \frac{\cos\alpha x}{x^2+1}dx = \left[\frac{\sin(\alpha x)}{\alpha(x^2+1)}\right]_{-\infty}^{\infty} + \frac{2}{\alpha}\int_{-\infty}^{\infty} \frac{x\sin(\alpha x)}{(x^2+1)^2}dx$$

$$= \frac{2}{\alpha}\int_{-\infty}^{\infty}\frac{x\sin(\alpha x)}{(x^2+1)^2}dx, \quad \alpha J(\alpha) = 2\int_{-\infty}^{\infty}\frac{x\sin(\alpha x)}{(x^2+1)^2}dx$$

양변을 미분하면

$$\frac{d}{d\alpha}(\alpha J(\alpha)) = J(\alpha)+\alpha J'(\alpha) = 2\int_{-\infty}^{\infty}\frac{\partial}{\partial \alpha}\frac{x\sin(\alpha x)}{(x^2+1)^2}dx = 2\int_{-\infty}^{\infty}\frac{x^2\cos(\alpha x)}{(x^2+1)^2}dx$$

$$= 2\int_{-\infty}^{\infty}\frac{\cos(\alpha x)}{x^2+1}dx - 2\int_{-\infty}^{\infty}\frac{\cos(\alpha x)}{(x^2+1)^2}dx = 2J(\alpha) - 2\int_{-\infty}^{\infty}\frac{\cos(\alpha x)}{(x^2+1)^2}dx$$

$\alpha J'(\alpha) - J(\alpha) = -2\int_{-\infty}^{\infty}\frac{\cos(\alpha x)}{(x^2+1)^2}dx$, 다시 양변을 α로 미분하면

$$\alpha J''(\alpha) = -2\int_{-\infty}^{\infty}\frac{\partial}{\partial \alpha}\frac{\cos(\alpha x)}{(x^2+1)^2}dx = 2\int_{-\infty}^{\infty}\frac{x\sin(\alpha x)}{(x^2+1)^2}dx = \alpha J(\alpha)$$

$J''(\alpha) = J(\alpha)$라는 미분방정식을 얻는다.

$J(\alpha)$의 이계도함수가 $J(\alpha)$와 동일해야 하므로 $J(\alpha)$는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4에 대하여
$J(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$와 $J(\alpha) = C_3\sin g(\alpha) + C_4\cos g(\alpha)$의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i\sin x$를 적용하면 두 경우가 동일함을 알 수 있다.)

$$J(\alpha) = C_1 e^{\alpha} + C_2 e^{-\alpha} \quad (\alpha \geq 0)$$

한편 $J(0) = C_1 + C_2 = \int_{-\infty}^{\infty}\frac{1}{x^2+1}dx = [\tan^{-1}x]_{-\infty}^{\infty} = \pi$,

$J(\alpha) = \frac{2}{\alpha}\int_{-\infty}^{\infty}\frac{x\sin(\alpha x)}{(x^2+1)^2}dx$에서 $\lim_{\alpha\to\infty}J(\alpha) = 0$, $C_1 = 0$이고 $C_2 = \pi$이다.

$$J(\alpha) = \int_{-\infty}^{\infty}\frac{\cos\alpha x}{x^2+1}dx = \frac{\pi}{e^{\alpha}}$$

$$\therefore I = 2\int_{-\infty}^{\infty}\frac{\cos t}{t^2+4}dt = 4\int_{-\infty}^{\infty}\frac{\cos 2u}{4u^2+4}du \ \ (t = 2u,\ dt = 2du)$$

$= J(2) = \dfrac{\pi}{e^2}$ ■

46. $I(\alpha) := \int_{-\infty}^{\infty} \dfrac{\cos\alpha x}{x^2+1} dx = \left[\dfrac{\sin(\alpha x)}{\alpha(x^2+1)} \right]_{-\infty}^{\infty} + \dfrac{2}{\alpha}\int_{-\infty}^{\infty} \dfrac{x\sin(\alpha x)}{(x^2+1)^2} dx$

$= \dfrac{2}{\alpha}\int_{-\infty}^{\infty} \dfrac{x\sin(\alpha x)}{(x^2+1)^2} dx$, $\quad \alpha I(\alpha) = 2\int_{-\infty}^{\infty} \dfrac{x\sin(\alpha x)}{(x^2+1)^2} dx$

양변을 미분하면

$\dfrac{d}{d\alpha}(\alpha I(\alpha)) = I(\alpha) + \alpha I'(\alpha) = 2\int_{-\infty}^{\infty} \dfrac{\partial}{\partial\,\alpha} \dfrac{x\sin(\alpha x)}{(x^2+1)^2} dx = 2\int_{-\infty}^{\infty} \dfrac{x^2\cos(\alpha x)}{(x^2+1)^2} dx$

$= 2\int_{-\infty}^{\infty} \dfrac{\cos(\alpha x)}{x^2+1} dx - 2\int_{-\infty}^{\infty} \dfrac{\cos(\alpha x)}{(x^2+1)^2} dx = 2I(\alpha) - 2\int_{-\infty}^{\infty} \dfrac{\cos(\alpha x)}{(x^2+1)^2} dx$

$\alpha I'(\alpha) - I(\alpha) = -2\int_{-\infty}^{\infty} \dfrac{\cos(\alpha x)}{(x^2+1)^2} dx$, 다시 양변을 α로 미분하면

$\alpha I''(\alpha) = -2\int_{-\infty}^{\infty} \dfrac{\partial}{\partial\,\alpha} \dfrac{\cos(\alpha x)}{(x^2+1)^2} dx = 2\int_{-\infty}^{\infty} \dfrac{x\sin(\alpha x)}{(x^2+1)^2} dx = \alpha I(\alpha)$

$I''(\alpha) = I(\alpha)$라는 미분방정식을 얻는다.

$I(\alpha)$의 이계도함수가 $I(\alpha)$와 동일해야 하므로 $I(\alpha)$는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4에 대하여
$I(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$와 $I(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i\sin x$를 적용하면 두 경우가 동일함을 알 수 있다.)

$$I(\alpha) = C_1 e^{\alpha} + C_2 e^{-\alpha} \quad (\alpha \ge 0)$$

한편 $I(0) = C_1 + C_2 = \int_{-\infty}^{\infty} \dfrac{1}{x^2+1} dx = [\tan^{-1} x]_{-\infty}^{\infty} = \pi$,

$I(\alpha) = \dfrac{2}{\alpha}\int_{-\infty}^{\infty} \dfrac{x\sin(\alpha x)}{(x^2+1)^2} dx$에서 $\lim_{\alpha\to\infty} I(\alpha) = 0$, $C_1 = 0$이고 $C_2 = \pi$이다.

$$I(\alpha) = \frac{\pi}{e^{\alpha}}, \quad \therefore \int_0^{\infty} \frac{\cos x}{x^2+1} dx = \frac{1}{2} I(1) = \frac{\pi}{2e} \ \blacksquare$$

47. $I(\alpha) := \int_0^{\infty} \frac{x \sin(\alpha x)}{x^2+1} dx = \int_0^{\infty} \frac{x^2 \sin(\alpha x)}{x(x^2+1)} dx \ (\alpha > 0)$

$$= \int_0^{\infty} \frac{(1+x^2-1)\sin(\alpha x)}{x(x^2+1)} dx = \int_0^{\infty} \frac{\sin(\alpha x)}{x} dx - \int_0^{\infty} \frac{\sin(\alpha x)}{x(1+x^2)} dx$$

$$= \int_0^{\infty} \frac{\sin t}{t} dt - \int_0^{\infty} \frac{\sin(\alpha x)}{x(1+x^2)} dx \ (t = \alpha x, \ dt = \alpha dx)$$

$$= \frac{\pi}{2} - \int_0^{\infty} \frac{\sin(\alpha x)}{x(1+x^2)} dx \ (※)$$

$$\frac{d}{d\alpha} I(\alpha) = -\int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\sin(\alpha x)}{x(1+x^2)} dx = -\int_0^{\infty} \frac{\cos(\alpha x)}{1+x^2} dx$$

$$\frac{d^2}{d\alpha^2} I(\alpha) = -\int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\cos(\alpha x)}{1+x^2} dx = \int_0^{\infty} \frac{x \sin(\alpha x)}{1+x^2} dx = I(\alpha)$$

$I''(\alpha) = I(\alpha)$라는 미분방정식을 얻는다.

$I(\alpha)$의 이계도함수가 $I(\alpha)$와 동일해야 하므로 $I(\alpha)$는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4에 대하여
$I(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$와 $I(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i \sin x$를 적용하면 두 경우가 동일함을 알 수 있다.)

$$I(\alpha) = C_1 e^{\alpha} + C_2 e^{-\alpha} \quad (\alpha > 0)$$

한편

$$\lim_{\alpha \to 0+} I(\alpha) = \lim_{\alpha \to 0+} \left(\frac{\pi}{2} - \int_0^{\infty} \frac{\sin(\alpha x)}{x(1+x^2)} dx \right) = \frac{\pi}{2}\text{에서 } C_1 + C_2 = \frac{\pi}{2},$$

$$\lim_{\alpha \to 0+} I'(\alpha) = \lim_{\alpha \to 0+} \left(-\int_0^{\infty} \frac{\cos(\alpha x)}{1+x^2} dx \right) = -\frac{\pi}{2}\text{에서 } C_1 - C_2 = -\frac{\pi}{2}\text{를 얻는다.}$$

따라서 $C_1=0$, $C_2=\dfrac{\pi}{2}$이고

$$I(\alpha)=\int_0^\infty \frac{x\sin(\alpha x)}{x^2+1}dx=\frac{\pi}{2e^\alpha},$$

$$\therefore \int_0^\infty \frac{x\sin x}{x^2+1}dx=I(1)=\frac{\pi}{2e} \ ■$$

※ $J(\alpha):=\displaystyle\int_0^\infty \frac{\sin x}{x}e^{-\alpha x}dx$, $\displaystyle\frac{d}{d\alpha}J(\alpha)=\int_0^\infty \frac{\partial}{\partial\alpha}\frac{\sin x}{x}e^{-\alpha x}dx=-\int_0^\infty e^{-\alpha x}\sin x dx$

$$=-\left[-e^{-\alpha x}\cos x\right]_0^\infty+\int_0^\infty \alpha e^{-\alpha x}\cos x dx=-1+\left[\alpha e^{-\alpha x}\sin x\right]_0^\infty+\int_0^\infty \alpha^2 e^{-\alpha x}\sin x dx$$

$$=-1-\alpha^2 J'(\alpha),\quad J'(\alpha)=-\frac{1}{1+\alpha^2}$$

$J(\alpha)=\displaystyle\int J'(\alpha)d\alpha=C-\tan^{-1}\alpha\ (C\in\mathbb{R})$, $\displaystyle 0=\lim_{\alpha\to\infty}J(\alpha)=C-\frac{\pi}{2}$에서 $C=\dfrac{\pi}{2}$이다.

$$\therefore \int_0^\infty \frac{\sin x}{x}dx=J(0)=\frac{\pi}{2} \ ■$$

48. $\displaystyle\int_0^\infty \frac{\ln(1+x)}{x(x^2+1)}dx=\int_0^1 \frac{\ln(1+x)}{x(x^2+1)}dx+\int_1^\infty \frac{\ln(1+x)}{x(x^2+1)}dx$

$$=\int_0^1 \frac{\ln(1+x)}{x(x^2+1)}dx+\int_1^0 \frac{t\ln(1+t^{-1})}{t^{-2}+1}\left(-\frac{1}{t^2}dt\right)\ \left(t=\frac{1}{x},\ dt=-\frac{1}{x^2}dx\right)$$

$$=\int_0^1 \frac{\ln(1+x)}{x(x^2+1)}dx+\int_0^1 \frac{t\ln(1+t^{-1})}{1+t^2}dt$$

$$=\int_0^1 \frac{\ln(1+x)}{x(x^2+1)}dx+\int_0^1 \frac{t\ln(1+t)}{1+t^2}dt-\int_0^1 \frac{t\ln t}{1+t^2}dt$$

$$=\int_0^1\left(t+\frac{1}{t}\right)\frac{\ln(1+t)}{1+t^2}dt-\int_0^1 \frac{t\ln t}{1+t^2}dt$$

$$=\int_0^1 \frac{\ln(1+t)}{t}dt-\frac{1}{4}\int_0^1 \frac{\ln u}{1+u}du\ \ (u=t^2,\ du=2tdt)$$

$$=\int_0^1 \frac{\ln(1+t)}{t}dt-\frac{1}{4}\left([\ln t\ln(1+t)]_0^1-\int_0^1 \frac{\ln(1+t)}{t}dt\right)=\frac{5}{4}\int_0^1 \frac{\ln(1+t)}{t}dt$$

$$=\frac{5}{4}\int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n u^n}{n+1}du=\frac{5}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\int_0^1 u^n du=\frac{5}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}$$

$$=\frac{5}{4}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\right)=\frac{5}{8}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{5}{8}\zeta(2)=\frac{5\pi^2}{48}\ ■$$

($\because$ 바젤 문제에 의해 $\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$)

※ $x=\tan t$의 치환을 한 후 $I(\alpha):=\int_0^{\frac{\pi}{2}}\frac{\ln(1+\alpha\tan t)}{\tan t}dt$로 정의하면 파인만 테크닉을 적용하여

$$I(\alpha)=\frac{\pi+2\alpha\ln\alpha}{2(1+\alpha^2)},\quad I(1)=\int_0^{\infty}\frac{\ln(1+x)}{x(1+x^2)}dx=\frac{5\pi^2}{48}$$

을 얻을 수 있다.

49. $v=\tan u$, $dv=\sec^2 u du$

$$I(\alpha):=\int_0^{\frac{\pi}{2}}\frac{\sin(u+\alpha\tan u)}{\sin u}du=\int_0^{\frac{\pi}{2}}\left(\cos(\alpha\tan u)+\frac{\sin(\alpha\tan u)}{\tan u}\right)du$$

$$=\int_0^{\infty}\frac{\cos(\alpha v)-\frac{\sin(\alpha v)}{v}}{1+v^2}dv,$$

$$\frac{d}{d\alpha}I(\alpha)=\int_0^{\infty}\frac{\partial}{\partial\alpha}\frac{\cos(\alpha v)-\frac{\sin(\alpha v)}{v}}{1+v^2}dv=\int_0^{\infty}\frac{\cos(\alpha v)-v\sin(\alpha v)}{1+v^2}dv$$

$$=\int_0^{\infty}\frac{\cos(\alpha v)}{1+v^2}dv-\int_0^{\infty}\frac{v\sin(\alpha v)}{1+v^2}dv=\frac{\pi}{2e}-\frac{\pi}{2e}=0\ (\because\ \#46,\ \#47)$$

$$\therefore I(\alpha)=C=I(0)=\frac{\pi}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin(u+\sqrt{\pi}\tan u)}{\sin u} du = I(\sqrt{\pi}) = \frac{\pi}{2} \ \blacksquare$$

50. $I(\alpha) := \int_0^{\alpha} \sin^2\left(x-\sqrt{\alpha^2-x^2}\right)dx = \int_{\frac{\pi}{2}}^{0} \sin^2(\alpha\cos t - \alpha\sin t)\cdot(-\alpha\sin t dt)$

$$= \int_0^{\frac{\pi}{2}} \sin^2(\alpha\cos t - \alpha\sin t)\alpha\sin t dt \ \ (x=\alpha\cos t,\ dx = -\alpha\sin t dt)$$

$$I(\alpha) := \int_0^{\alpha} \sin^2\left(x-\sqrt{\alpha^2-x^2}\right)dx = \int_0^{\frac{\pi}{2}} \sin^2(\alpha\sin t - \alpha\cos t)\alpha\cos t dt$$

$$= \int_0^{\frac{\pi}{2}} \sin^2(\alpha\cos t - \alpha\sin t)\alpha\cos t dt \ \ (x=\alpha\sin t,\ dx = \alpha\cos t dt)$$

$$I(\alpha) = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2(\alpha(\sin t - \cos t))\alpha(\sin t + \cos t)dt$$

$$= \frac{1}{2}\int_{-\alpha}^{\alpha} \sin^2 u du \ \ (u=\alpha(\sin t - \cos t),\ du = \alpha(\cos t + \sin t)dt)$$

$$= \frac{1}{4}\int_{-\alpha}^{\alpha} (1-\cos 2u)du = \frac{1}{4}\left[u - \frac{1}{2}\sin 2u\right]_{-\alpha}^{\alpha} = \frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha$$

$$\therefore \int_0^{\pi} \sin^2\left(x-\sqrt{\pi^2-x^2}\right)dx = I(\pi) = \frac{\pi}{2} \ \blacksquare$$

51. $I(\alpha) := \int_0^{\infty} e^{-x^2}\cos(\alpha x)dx$

$$\frac{d}{d\alpha}I(\alpha) = \int_0^{\infty} \frac{\partial}{\partial\alpha} e^{-x^2}\cos(\alpha x)dx = -\int_0^{\infty} xe^{-x^2}\sin(\alpha x)dx$$

$$= -\left(\left[-\frac{1}{2}e^{-x^2}\sin(\alpha x)\right]_0^{\infty} + \frac{\alpha}{2}\int_0^{\infty} e^{-x^2}\cos(\alpha x)dx\right) = -\frac{\alpha}{2}I(\alpha)$$

$I'(\alpha) = -\frac{\alpha}{2}I(\alpha)$의 미분방정식을 얻는다. 이를 풀기 위해 변수를 분리하고 양변을 적분하면

$$\frac{1}{I(\alpha)}dI(\alpha) = -\frac{\alpha}{2}d\alpha, \quad \int \frac{1}{I(\alpha)}dI(\alpha) = -\int \frac{\alpha}{2}d\alpha, \quad \ln|I(\alpha)| + C = -\frac{1}{4}\alpha^2$$

$I(0) = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ ($\because$ 가우스 적분) 이므로 $C = -\ln\left(\frac{\sqrt{\pi}}{2}\right)$

$$I(\alpha) = \int_0^\infty e^{-x^2}\cos(\alpha x)dx = \frac{\sqrt{\pi}}{2}e^{-\frac{\alpha^2}{4}}$$

$$\therefore \int_0^\infty e^{-x^2}\cos 2x dx = I(2) = \frac{\sqrt{\pi}}{2e} \ \blacksquare$$

52. $I(\alpha) := \int_0^{\frac{\pi}{2}} \cos(\alpha \tan x)dx$, $t = \tan x$, $dt = \sec^2 x dx$, $dx = \dfrac{dt}{1+t^2}$

$$I(\alpha) = \int_0^{\frac{\pi}{2}} \cos(\alpha \tan x)dx = \int_0^\infty \frac{\cos\alpha t}{1+t^2}dt = \frac{1}{\alpha}\left[\frac{\sin\alpha t}{1+t^2}\right]_0^\infty + \frac{1}{\alpha}\int_0^\infty \frac{2t\sin\alpha t}{(1+t^2)^2}dt$$

$= \dfrac{1}{\alpha}\displaystyle\int_0^\infty \frac{2t\sin\alpha t}{(1+t^2)^2}dt$, $\alpha I(\alpha)$에 대하여 파인만 적분 테크닉을 이용하면

$$\frac{d}{d\alpha}(\alpha I(\alpha)) = I(\alpha) + \alpha I'(\alpha) = \int_0^\infty \frac{\partial}{\partial \alpha}\left(\frac{2t\sin\alpha t}{(1+t^2)^2}\right)dt = \int_0^\infty \frac{2t^2\cos\alpha t}{(1+t^2)^2}dt$$

$$= \int_0^\infty \frac{2(t^2+1-1)\cos\alpha t}{(1+t^2)^2}dt = 2\int_0^\infty \frac{\cos\alpha t}{1+t^2}dt - \int_0^\infty \frac{2\cos\alpha t}{(1+t^2)^2}dt$$

$$= 2I(\alpha) - \int_0^\infty \frac{2\cos\alpha t}{(1+t^2)^2}dt.$$

다시 양변을 α에 대하여 미분하면

$$I'(\alpha) + \alpha I''(\alpha) + I'(\alpha) = 2I'(\alpha) - \int_0^\infty \frac{\partial}{\partial \alpha}\frac{2\cos\alpha t}{(1+t^2)^2}dt = 2I'(\alpha) + \int_0^\infty \frac{2t\sin\alpha t}{(1+t^2)^2}dt$$

$\alpha I''(\alpha) = \displaystyle\int_0^\infty \frac{2t\sin\alpha t}{(1+t^2)^2}dt = \alpha I(\alpha)$, $\quad I''(\alpha) = I(\alpha)$라는 미분방정식을 얻는다.

$I(\alpha)$의 이계도함수가 $I(\alpha)$와 동일해야 하므로 $I(\alpha)$는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 C_1, C_2, C_3, C_4에 대하여

$I(\alpha)=C_1e^{f(\alpha)}+C_2e^{-f(\alpha)}$와 $I(\alpha)=C_3\sin g(\alpha)+C_4\cos g(\alpha)$의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix}=\cos x+i\sin x$를 적용하면 두 경우가 동일함을 알 수 있다.)

$$I(\alpha)=C_1e^{\alpha}+C_2e^{-\alpha}\quad(\alpha\ge 0)$$

이때 $I(\alpha)$는 $\cos(\alpha\tan x)$를 0부터 $\frac{\pi}{2}$까지 정적분한 값이므로 모든 $\alpha\ge 0$에 대하여 유계(bounded)이고, 따라서 $C_1=0$이다. 또한 $\alpha=0$을 대입하면 $I(0)=\frac{\pi}{2}=C_2$를 얻는다.

$$I(\alpha)=\int_0^{\frac{\pi}{2}}\cos(\alpha\tan x)dx=\frac{\pi}{2e^{\alpha}},$$

$$\therefore\int_0^{\frac{\pi}{2}}\cos(\tan x)dx=I(1)=\frac{\pi}{2e}\ ■$$

53. $I(\alpha):=\displaystyle\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(\alpha\tan x)}{\tan x}dx$

$$\frac{d}{d\alpha}I(\alpha)=\int_0^{\frac{\pi}{2}}\frac{\partial}{\partial\alpha}\frac{\tan^{-1}(\alpha\tan x)}{\tan x}dx=\int_0^{\frac{\pi}{2}}\frac{1}{1+\alpha^2\tan^2x}dx$$

$$=\int_0^{\infty}\frac{1}{(1+\alpha^2t^2)(1+t^2)}dt\ \ (t=\tan x,\ dt=\sec^2x dx)$$

$$=\frac{1}{\alpha^2-1}\int_0^{\infty}\left(\frac{\alpha^2}{1+\alpha^2t^2}-\frac{1}{1+t^2}\right)dt=\frac{1}{\alpha^2-1}\left[\alpha\tan^{-1}(\alpha t)-\tan^{-1}t\right]_0^{\infty}=\frac{\pi}{2(\alpha+1)}$$

$I(\alpha)=\frac{\pi}{2}\ln|\alpha+1|+C,\quad C=I(0)=0$에서 $I(\alpha)=\frac{\pi}{2}\ln|\alpha+1|$

$$\therefore\int_0^{\frac{\pi}{2}}\frac{x}{\tan x}dx=I(1)=\frac{\pi}{2}\ln 2\ ■$$

54. $I(\alpha):=\displaystyle\int_0^{\pi}\ln(1-2\alpha\cos x+\alpha^2)dx,\ \frac{d}{d\alpha}I(\alpha)=\int_0^{\pi}\frac{\partial}{\partial\alpha}\ln(1-2\alpha\cos x+\alpha^2)dx$

$$=\int_0^{\pi}\frac{2(\alpha-\cos x)}{1-2\alpha\cos x+\alpha^2}dx=\frac{1}{\alpha}\int_0^{\pi}\frac{2(\alpha^2-\alpha\cos x)}{1-2\alpha\cos x+\alpha^2}dx$$

$$=\frac{1}{\alpha}\int_0^{\pi}\left(\frac{2(\alpha^2-\alpha\cos x)}{1-2\alpha\cos x+\alpha^2}-1+1\right)dx$$

$$=\frac{1}{\alpha}\int_0^{\pi}\left(\frac{2(\alpha^2-\alpha\cos x)}{1-2\alpha\cos x+\alpha^2}-\frac{1-2\alpha\cos x+\alpha^2}{1-2\alpha\cos x+\alpha^2}+1\right)dx$$

$$=\frac{1}{\alpha}\int_0^{\pi}\left(1-\frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos x}\right)dx=\frac{\pi}{\alpha}-\frac{1-\alpha^2}{\alpha(1+\alpha^2)}\int_0^{\pi}\left(\frac{1}{1-\dfrac{2\alpha}{1+\alpha^2}\cos x}\right)dx$$

$$=\frac{\pi}{\alpha}-\frac{1-\alpha^2}{\alpha(1+\alpha^2)}\int_0^{\infty}\frac{2}{(1+t^2)\left(1-\dfrac{2\alpha}{1+\alpha^2}\dfrac{1-t^2}{1+t^2}\right)}dt\ \ \left(t=\tan\frac{x}{2},\ dt=\frac{1}{2}\sec^2\frac{x}{2}dx\right)$$

$$=\frac{\pi}{\alpha}-\frac{1-\alpha^2}{\alpha(1+\alpha^2)}\int_0^{\infty}\frac{2}{\left(1-\dfrac{2\alpha}{1+\alpha^2}\right)+\left(1+\dfrac{2\alpha}{1+\alpha^2}\right)t^2}dt$$

$$=\frac{\pi}{\alpha}-\frac{1-\alpha^2}{\alpha(1+\alpha^2)}\int_0^{\infty}\frac{2(1+\alpha^2)}{(1+\alpha^2-2\alpha)+(1+\alpha^2+2\alpha)t^2}dt$$

$$=\frac{\pi}{\alpha}-\frac{1-\alpha^2}{\alpha(1+\alpha^2)}\int_0^{\infty}\frac{2(1+\alpha^2)}{(1-\alpha)^2+(1+\alpha)^2t^2}dt=\frac{\pi}{\alpha}-\frac{2(1+\alpha)}{\alpha(1-\alpha)}\int_0^{\infty}\frac{1}{1+\left(\dfrac{\alpha+1}{\alpha-1}\right)^2t^2}dt$$

$$=\frac{\pi}{\alpha}-\frac{2(1+\alpha)}{\alpha(1-\alpha)}\left[\frac{\alpha-1}{\alpha+1}\tan^{-1}\left(\frac{\alpha+1}{\alpha-1}\right)t\right]_0^{\infty}=\frac{\pi}{\alpha}+\frac{\pi}{\alpha}=\frac{2\pi}{\alpha}$$

$$I(\alpha)=\int\frac{2\pi}{\alpha}d\alpha=2\pi\ln|\alpha|+C,\ \ I(1)=\int_0^{\pi}\ln(2-2\cos x)dx=0=C\ (※)$$

$$I(\alpha)=\int_0^{\pi}\ln(1-2\alpha\cos x+\alpha^2)dx=2\pi\ln|\alpha|$$

$$\therefore\int_0^{\pi}\ln(1-2e\cos x+e^2)dx=I(e)=2\pi\ \blacksquare$$

$$※\ \ J=\int_0^{\pi}\ln(2-2\cos x)dx=\pi\ln 2+\int_0^{\pi}\ln(1-\cos x)dx$$

$$=\pi\ln2+\int_0^{\pi}\ln(1+\cos x)dx \;\; (x\mapsto \pi-x)$$

$$=\pi\ln2+\frac{1}{2}\int_0^{\pi}\ln(1-\cos^2 x)dx=\pi\ln2+\frac{1}{2}\int_0^{\pi}\ln(\sin^2 x)dx$$

$$=\pi\ln2+\int_0^{\pi}\ln(\sin x)dx=\pi\ln2+2K$$

$$K=\frac{1}{2}\int_0^{\pi}\ln(\sin x)dx=\int_0^{\frac{\pi}{2}}\ln(\sin x)dx=\int_0^{\frac{\pi}{2}}\ln(\cos x)dx \;\; \left(x\mapsto \frac{\pi}{2}-x\right)$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin x\cos x)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\left(\frac{1}{2}\sin 2x\right)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin 2x)dx-\frac{\pi}{4}\ln2$$

$$=\frac{1}{4}\int_0^{\pi}\ln(\sin t)dt-\frac{\pi}{4}\ln2 \;\; (t=2x,\; dt=2dx)$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin t)dt-\frac{\pi}{4}\ln2=\frac{1}{2}I-\frac{\pi}{4}\ln2,\quad \frac{1}{2}I=-\frac{\pi}{4}\ln2$$

$$\therefore K=\int_0^{\frac{\pi}{2}}\ln(\sin x)dx=-\frac{\pi}{2}\ln2$$

$$\therefore J=\pi\ln2+2K=0 \;\blacksquare$$

55. $x=\tan t$, $dx=\sec^2 t dt$

$$I=\int_0^{\pi}\ln(1+\sin^2 t)dt=2\int_0^{\frac{\pi}{2}}\ln(1+\sin^2 t)dt=2\int_0^{\infty}\frac{\ln(2+x^2)-\ln(1+x^2)}{1+x^2}dx$$

$$I(\alpha):=\int_0^{\infty}\frac{\ln(\alpha+x^2)-\ln(1+x^2)}{1+x^2}dx,$$

$$\frac{d}{d\alpha}I(\alpha)=\int_0^{\infty}\frac{\partial}{\partial\alpha}\frac{\ln(\alpha+x^2)-\ln(1+x^2)}{1+x^2}dx=\int_0^{\infty}\frac{1}{(1+x^2)(\alpha+x^2)}dx$$

$$=\frac{1}{\alpha-1}\left(\int_0^{\infty}\frac{1}{x^2+1}dx-\int_0^{\infty}\frac{1}{x^2+\alpha}dx\right)=\frac{1}{\alpha-1}\left(\left[\tan^{-1}x\right]_0^{\infty}-\left[\frac{1}{\sqrt{\alpha}}\tan^{-1}\left(\frac{x}{\sqrt{\alpha}}\right)\right]_0^{\infty}\right)$$

$$= \frac{\pi}{2\sqrt{\alpha}(1+\sqrt{\alpha})}$$

한편 $I(1) = \int_0^{\infty} 0\,dx = 0$이므로

$$I = 2I(2) = 2\int_1^2 \frac{d}{d\alpha} I(\alpha) d\alpha = \pi \int_1^2 \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})} d\alpha$$

$$= 2\pi \int_1^{\sqrt{2}} \frac{1}{1+u} du \ \ (\alpha = u^2,\ d\alpha = 2udu)$$

$$= 2\pi [\ln|1+u|]_1^{\sqrt{2}} = 2\pi \ln\left(\frac{1+\sqrt{2}}{2}\right) \blacksquare$$

56. $x = u+2,\ dx = du$

$$I = \int_0^4 \frac{\ln x}{\sqrt{4x-x^2}} dx = \int_{-2}^2 \frac{\ln(u+2)}{\sqrt{4-u^2}} du = \int_0^2 \frac{\ln(u+2)}{\sqrt{4-u^2}} du + \int_{-2}^0 \frac{\ln(u+2)}{\sqrt{4-u^2}} du$$

$$= \int_0^2 \frac{\ln(u+2)}{\sqrt{4-u^2}} du + \int_2^0 \frac{\ln(-u+2)}{\sqrt{4-u^2}} (-du) \ \ (u \mapsto -u)$$

$$= \int_0^2 \frac{\ln(4-u^2)}{\sqrt{4-u^2}} du$$

한편 $x = 4-u^2,\ dx = -2udu$의 치환을 하면

$$I = \int_0^4 \frac{\ln x}{\sqrt{4x-x^2}} dx = \int_2^0 \frac{\ln(4-u^2)}{u\sqrt{4-u^2}} \cdot (-2udu) = 2\int_0^2 \frac{\ln(4-u^2)}{\sqrt{4-u^2}} du$$

$$I = 2I,\ \therefore I = \int_0^4 \frac{\ln x}{\sqrt{4x-x^2}} dx = 0 \ \blacksquare$$

57. $x = \tan t,\ dx = \sec^2 t dt,\ \frac{dx}{1+x^2} = dt$

$$I = \int_0^{\infty} \frac{1}{(1+x^2)(1+x^{\pi})} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^{\pi} t} dt$$

$$= \int_{\frac{\pi}{2}}^{0} \frac{1}{1+\tan^{\pi}\left(\frac{\pi}{2}-t\right)}(-dt) = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^{\pi}t} dt = \int_0^{\frac{\pi}{2}} \frac{\tan^{\pi}t}{1+\tan^{\pi}t} dt \ \left(t \mapsto \frac{\pi}{2}-t\right)$$

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{1}{1+\tan^{\pi}t} + \frac{\tan^{\pi}t}{1+\tan^{\pi}t}\right) dt = \frac{\pi}{2}$$

$$\therefore I = \int_0^{\infty} \frac{1}{(1+x^2)(1+x^{\pi})} dx = \frac{\pi}{4} \ \blacksquare$$

58. sol 1) $I(v) := \int_0^{\infty} \frac{1}{(v+x)(\pi^2+(\ln x)^2)} dx$, $J(v) := \int_{-\infty}^{\infty} \frac{1}{(v+e^t)(\pi^2+t^2)} dt$

$x = e^t$, $dx = e^t dt$

$$I(v) = \int_{-\infty}^{\infty} \frac{e^t}{(v+e^t)(\pi^2+t^2)} dt = \int_{-\infty}^{\infty} \frac{(v+e^t)}{(v+e^t)(\pi^2+t^2)} dt - v\int_{-\infty}^{\infty} \frac{1}{(v+e^t)(\pi^2+t^2)} dt$$

$$= \left[\frac{1}{\pi}\tan^{-1}\left(\frac{x}{\pi}\right)\right]_{-\infty}^{\infty} - vJ(v) = 1 - vJ(v)$$

$t = -z$, $dt = -dz$

$$I(v) = \int_{\infty}^{-\infty} \frac{e^{-z}}{(v+e^{-z})(\pi^2+z^2)}(-dz) = \int_{-\infty}^{\infty} \frac{1}{(1+ve^z)(\pi^2+z^2)} dz = \frac{1}{v} J\left(\frac{1}{v}\right)$$

$$1 - vJ(v) = I(v) = \frac{1}{v} J\left(\frac{1}{v}\right), \quad vJ(v) + \frac{1}{v} J\left(\frac{1}{v}\right) = 1$$

$$v = 1, \ \therefore J(1) = I(1) = \int_0^{\infty} \frac{1}{(1+x)(\pi^2+(\ln x)^2)} dx = \frac{1}{2} \ \blacksquare$$

sol 2) $I = \int_0^{\infty} \frac{1}{(1+x)(\pi^2+(\ln x)^2)} dx$

$$= \int_0^1 \frac{1}{(1+x)(\pi^2+(\ln x)^2)} dx + \int_1^{\infty} \frac{1}{(1+x)(\pi^2+(\ln x)^2)} dx$$

$$= \int_0^1 \frac{1}{(1+x)(\pi^2+(\ln x)^2)} dx + \int_1^0 \frac{1}{(1+1/x)(\pi^2+(\ln x)^2)} \cdot \left(-\frac{1}{x^2}\right) dx \ \left(x \mapsto \frac{1}{x}\right)$$

$$= \int_0^1 \frac{1}{(1+x)(\pi^2+(\ln x)^2)}dx + \int_0^1 \frac{1}{x(1+x)(\pi^2+(\ln x)^2)}dx$$

$$= \int_0^1 \left(\frac{1}{(1+x)(\pi^2+(\ln x)^2)} \cdot \frac{x+1}{x} \right)dx = \int_0^1 \frac{1}{x(\pi^2+(\ln x)^2)}dx$$

$$= \int_{-\infty}^0 \frac{1}{\pi^2+u^2}du \quad (u=\ln x,\ du=\frac{dx}{x})$$

$$= \left[\frac{1}{\pi}\tan^{-1}\left(\frac{u}{\pi}\right) \right]_{-\infty}^0 = \frac{1}{2} \ \blacksquare$$

59\. $I = \int_0^\infty \frac{x-1}{\sqrt{2^x-1}\ln(2^x-1)}dx$

$$= \frac{1}{(\ln 2)^2}\int_0^\infty \frac{\ln(t^2+1)-\ln 2}{(t^2+1)\ln t}dt \ \ (t=\sqrt{2^x-1},\ dt=\frac{2^{x-1}\ln 2}{\sqrt{2^x-1}}dx)$$

$$= \frac{1}{(\ln 2)^2}\int_\infty^0 \frac{\ln(u^{-2}+1)-\ln 2}{\left(\frac{1}{u^2}+1\right)(-\ln u)}\left(-\frac{1}{u^2}\right)du = -\frac{1}{(\ln 2)^2}\int_0^\infty \frac{\ln(u^2+1)-\ln u^2-\ln 2}{(u^2+1)\ln u}du$$

$(u=\frac{1}{t},\ dt=-\frac{1}{u^2}du)$

$$= -\frac{1}{(\ln 2)^2}\int_0^\infty \frac{\ln(u^2+1)-\ln 2}{(u^2+1)\ln u}du + \frac{2}{(\ln 2)^2}\int_0^\infty \frac{1}{u^2+1}du$$

$$= -I + \frac{2}{(\ln 2)^2}\left[\tan^{-1}u\right]_0^\infty = -I + \frac{\pi}{(\ln 2)^2}$$

$$\therefore I = \int_0^\infty \frac{x-1}{\sqrt{2^x-1}\ln(2^x-1)}dx = \frac{\pi}{2(\ln 2)^2} \ \blacksquare$$

60\. $t=\tan x,\ dt=\sec^2 x dx$

$$\int_0^{3\pi} \frac{1}{\sin^4 x+\cos^4 x}dx = 6\int_0^{\frac{\pi}{2}} \frac{\sec^4 x}{1+\tan^4 x}dx = 6\int_0^\infty \frac{1+t^2}{1+t^4}dt$$

$$= 6\int_0^\infty \left(\frac{1}{2(t^2+\sqrt{2}t+1)} - \frac{1}{2(-t^2+\sqrt{2}t-1)} \right)dt$$

$$= 3\int_0^\infty \frac{1}{t^2+\sqrt{2}t+1}dt + 3\int_0^\infty \frac{1}{t^2-\sqrt{2}t+1}dt$$

$$= 3\int_0^\infty \frac{1}{\left(t+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}dt + 3\int_0^\infty \frac{1}{\left(t-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}dt$$

$$= 3\left[\sqrt{2}\tan^{-1}(\sqrt{2}t+1)\right]_0^\infty + 3\left[\sqrt{2}\tan^{-1}(\sqrt{2}t-1)\right]_0^\infty = 3\sqrt{2}\pi \ \blacksquare$$

61. $\int_0^\pi \cos^4 x dx = \frac{1}{4}\int_0^\pi (1-\cos 2x)^2 dx = \frac{1}{4}\int_0^\pi \left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)dx$

$$= \int_0^\pi \left(\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\right)dx = \left[\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x\right]_0^\pi = \frac{3}{8}\pi \ \blacksquare$$

62. $\int_0^{\frac{\pi}{2}} \sin^6 x\cos^3 x dx = \int_0^{\frac{\pi}{2}} \sin^6 x(1-\sin^2 x)\cos x dx$

$$= \int_0^{\frac{\pi}{2}} \sin^6 x\cos x dx - \int_0^{\frac{\pi}{2}} \sin^8 x\cos x dx = \left[\frac{1}{7}\sin^7 x-\frac{1}{9}\sin^9 x\right]_0^{\frac{\pi}{2}} = \frac{2}{63} \ \blacksquare$$

63. $\int_0^{\frac{\pi}{4}} \sin^3 x\sec^2 x dx = \int_0^{\frac{\pi}{4}} \frac{\sin x(1-\cos^2 x)}{\cos^2 x}dx = \int_0^{\frac{\pi}{4}} (\tan x\sec x-\sin x)dx$

$$= [\sec x+\cos x]_0^{\frac{\pi}{4}} = \frac{3\sqrt{2}}{2}-2 \ \blacksquare$$

64. $x = 2\tan t$, $dx = 2\sec^2 t dt$

$$\int_1^3 \frac{1}{x^2\sqrt{x^2+4}}dx = \int_\alpha^\beta \frac{\sec t}{4\tan^2 t}dt = \frac{1}{4}\int_\alpha^\beta \csc t\cot t dt = \left[-\frac{1}{4}\csc t\right]_\alpha^\beta = \left[-\frac{\sqrt{x^2+4}}{4x}\right]_1^3$$

$$= \frac{3\sqrt{5}-\sqrt{13}}{12} \ \blacksquare \quad \left(\alpha = \tan^{-1}\left(\frac{1}{2}\right),\ \beta = \tan^{-1}\left(\frac{3}{2}\right)\right)$$

65. $x+1 = \tan t$, $dx = \sec^2 t dt$

$$\int_{\sqrt{3}-1}^{2\sqrt{2}-1}\frac{1}{(x+1)\sqrt{x^2+2x+2}}dx=\int_{\alpha}^{\beta}\frac{\sec^2 t}{\tan t\sec t}dt=\int_{\alpha}^{\beta}\csc t dt=[\ln|\csc t-\cot t|]_{\alpha}^{\beta}$$

$$=\left[\ln\left|\frac{\sqrt{x^2+2x+2}-1}{x+1}\right|\right]_{\sqrt{3}-1}^{2\sqrt{2}-1}=\ln\sqrt{\frac{3}{2}}\ \blacksquare\quad \left(\alpha=\frac{\pi}{3},\ \beta=\tan^{-1}(2\sqrt{2})\right)$$

66. $$\int_0^{\tan^{-1}(\sqrt{6})}\frac{1}{1+\cos^2 x}dx=\int_0^{\tan^{-1}(\sqrt{6})}\sec^2 x\cdot\frac{1}{\tan^2 x+2}dx$$

$$=\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)\right]_0^{\tan^{-1}(\sqrt{6})}=\frac{\sqrt{2}}{6}\pi\ \blacksquare$$

67. sol 1) $$\int_0^{\frac{\pi}{2}}\frac{1}{1+\cos x}dx=\int_0^{\frac{\pi}{2}}\frac{1-\cos^2 x}{\sin^2 x}dx=\int_0^{\frac{\pi}{2}}(\csc^2 x-\csc x\cot x)dx$$

$$=[\csc x-\cot x]_0^{\frac{\pi}{2}}=1-\lim_{x\to 0}(\csc x-\cot x)=1\ \blacksquare$$

sol 2) $$\int_0^{\frac{\pi}{2}}\frac{1}{1+\cos x}dx=\int_0^{\frac{\pi}{2}}\frac{1}{2\cos^2\frac{x}{2}}dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sec^2\frac{x}{2}dx=\left[\tan\frac{x}{2}\right]_0^{\frac{\pi}{2}}=1\ \blacksquare$$

68. $t=e^x+1$, $dt=e^x dx$

$$\int_{\ln(e-1)}^{\ln(e^3-1)}\frac{e^x\ln(e^x+1)}{e^x+1}dx=\int_e^{e^3}\frac{\ln t}{t}dt=\left[\frac{1}{2}(\ln t)^2\right]_e^{e^3}=4\ \blacksquare$$

69. $$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1}{\sin x}dx=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\csc x dx=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\csc^2 x-\csc x\cot x}{\csc x-\cot x}dx$$

$$=[\ln|\csc x-\cot x|]_{\pi/3}^{\pi/2}=\ln\sqrt{3}\ \blacksquare$$

70. $$\int_0^{\frac{\pi}{4}}\frac{1}{1-3\cos^2 x}dx=\int_0^{\frac{\pi}{4}}\sec^2 x\cdot\frac{1}{\tan^2 x-2}dx=\left[\frac{1}{2\sqrt{2}}\ln\left|\frac{\tan x-\sqrt{2}}{\tan x+\sqrt{2}}\right|\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{2\sqrt{2}}\ln(3-2\sqrt{2})=\frac{1}{\sqrt{2}}\ln(\sqrt{2}-1)\ \blacksquare$$

71. $\int_{-1}^{0} \frac{x^3-x-2}{x^3-x^2+x-1} dx = \int_{-1}^{0} \left(1-\frac{1}{x-1}+\frac{2x}{x^2+1}\right) dx = \left[x-\ln|x-1|+\ln(x^2+1)\right]_{-1}^{0}$

$= 1$ ■

72. $\int_{3}^{4} \frac{x+4}{x^3+3x^2-10x} dx = \int_{3}^{4} \frac{x+4}{x(x+5)(x-2)} dx = \int_{3}^{4} \left(-\frac{2}{5x}-\frac{1}{35(x+5)}+\frac{3}{7(x-2)}\right) dx$

$= \left[-\frac{2}{5}\ln|x|-\frac{1}{35}\ln|x+5|+\frac{3}{7}\ln|x-2|\right]_{3}^{4} = \frac{12}{35}\ln 3-\frac{10}{35}\ln 2 = \frac{2}{35}\ln\left(\frac{729}{32}\right)$ ■

73. $\int_{5}^{6} \frac{7x^3-13x^2-24x+24}{x^4-3x^3-10x^2+24x} dx = \int_{5}^{6} \frac{7x^3-13x^2-24x+24}{x(x-2)(x+3)(x-4)} dx$

$= \int_{5}^{6} \left(\frac{1}{x-2}+\frac{1}{x}+\frac{2}{x+3}+\frac{3}{x-4}\right) dx = \left[\ln|x-2|+\ln|x|+2\ln|x+3|+3\ln|x-4|\right]_{5}^{6}$

$= 4\ln 3-\ln 5 = \ln\left(\frac{81}{5}\right)$ ■

74. $t = \frac{1}{x}$, $dt = -\frac{1}{x^2} dx$

$$\int_{1}^{2} \frac{1}{x^2\sqrt{2x-x^2}} dx = \int_{1}^{2} \frac{1}{x^3\sqrt{\frac{2}{x}-1}} dx = -\int_{1}^{\frac{1}{2}} \frac{t}{\sqrt{2t-1}} dt = \frac{1}{2}\int_{\frac{1}{2}}^{1} \frac{2t-1+1}{\sqrt{2t-1}} dt$$

$= \frac{1}{2}\int_{\frac{1}{2}}^{1} \left(\sqrt{2t-1}+\frac{1}{\sqrt{2t-1}}\right) dt = \frac{1}{2}\left[\frac{1}{6}(2t-1)^{\frac{3}{2}}+\frac{1}{2}\sqrt{2t-1}\right]_{\frac{1}{2}}^{1} = \frac{2}{3}$ ■

75. $t = x^2$, $dt = 2x dx$

$$\int_{1}^{e} \frac{x^4+81}{x(x^2+9)^2} dx = \frac{1}{2}\int_{1}^{e^2} \frac{t^2+81}{t(t+9)^2} dt = \frac{1}{2}\int_{1}^{e^2} \left(\frac{1}{t}-\frac{18}{(t+9)^2}\right) dt$$

$= \left[\frac{1}{2}\ln|t|+\frac{9}{t+9}\right]_{1}^{e^2} = \frac{e^2+99}{10(e^2+9)}$ ■

76. $t = \sqrt{x}$, $dt = \frac{1}{2\sqrt{x}} dx$, $t = 2\sin\theta$, $dt = 2\cos\theta d\theta$, $\sin\theta = \frac{\sqrt{x}}{2}$

$$\int_0^3 \sqrt{\frac{4-x}{x}}\,dx = \int_0^{\sqrt{3}} \frac{\sqrt{4-t^2}}{t} \cdot 2tdt = 2\int_0^{\sqrt{3}} \sqrt{4-t^2}\,dt = 2\int_0^{\frac{\pi}{3}} 4\cos^2\theta d\theta$$

$$= 4\int_0^{\frac{\pi}{3}} (1+\cos 2\theta)d\theta = [4\theta + 4\sin\theta\cos\theta]_0^{\frac{\pi}{3}} = \sqrt{3} + \frac{4}{3}\pi \ ■$$

77. $t = x^{\frac{3}{2}}$, $dt = \frac{3}{2}\sqrt{x}\,dx$

$$\int_0^{\frac{1}{\sqrt[3]{2}}} \sqrt{\frac{x}{1-x^3}}\,dx = \frac{2}{3}\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-t^2}}\,dt = \left[\frac{2}{3}\sin^{-1}t\right]_0^{\frac{1}{\sqrt{2}}} = \frac{\pi}{6} \ ■$$

78. $t = \sqrt{x}$, $dt = \frac{1}{2\sqrt{x}}dx$, $t = \sin u$, $dt = \cos u du$

$$\int_0^{\frac{3}{4}} \sqrt{x}\sqrt{1-x}\,dx = \int_0^{\frac{\sqrt{3}}{2}} t\sqrt{1-t^2} \cdot 2tdt = 2\int_0^{\frac{\sqrt{3}}{2}} t^2\sqrt{1-t^2}\,dt$$

$$= 2\int_0^{\frac{\pi}{3}} \sin^2 u\cos u \cdot \cos u du = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2 2u du = \frac{1}{4}\int_0^{\frac{\pi}{3}} (1-\cos 4u)du$$

$$= \left[\frac{1}{4}u - \frac{1}{16}\sin 4u\right]_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{32} + \frac{\pi}{12} \ ■$$

79. $t = \tan x$, $dt = \sec^2 x dx$

$$\int_0^{\tan^{-1}(\sqrt[4]{3})} \frac{\sin x\cos x}{\sin^4 x + \cos^4 x}dx = \int_0^{\tan^{-1}(\sqrt[4]{3})} \frac{\tan x\sec^2 x}{1+\tan^4 x}dx = \int_0^{\sqrt[4]{3}} \frac{t}{1+t^4}dt$$

$$= \left[\frac{1}{2}\tan^{-1}(t^2)\right]_0^{\sqrt[4]{3}} = \frac{\pi}{6} \ ■$$

80. $\displaystyle\int_0^{\frac{\pi}{4}} \tan^4 x\sec^4 x dx = \int_0^{\frac{\pi}{4}} \tan^4 x(\tan^2 x + 1)\sec^2 x dx$

$$= \int_0^{\frac{\pi}{4}} \tan^6 x\sec^2 x dx + \int_0^{\frac{\pi}{4}} \tan^4 x\sec^2 x dx = \left[\frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x\right]_0^{\frac{\pi}{4}} = \frac{12}{35} \ ■$$

81. $t=\sqrt{x}$, $dt=\dfrac{1}{2\sqrt{x}}dx$

$$\int_2^4 \frac{x\sqrt{x}-1}{x-\sqrt{x}}dx = \int_{\sqrt{2}}^2 \frac{t^3-1}{t^2-t}\cdot 2tdt = 2\int_{\sqrt{2}}^2 (t^2+t+1)dt = \left[\frac{2}{3}t^3+t^2+2t\right]_{\sqrt{2}}^2$$

$$= \frac{34-10\sqrt{2}}{3} \ \blacksquare$$

82. $t=e^x$, $dt=e^x dx$, $u=\sqrt{t+1}$, $du=\dfrac{1}{2\sqrt{t+1}}dt$

$$\int_0^{\ln 3} \frac{1}{\sqrt{1+e^x}}dx = \int_1^3 \frac{1}{t\sqrt{t+1}}dt = \int_{\sqrt{2}}^2 \frac{2}{u^2-1}du = \left[\ln\left|\frac{u-1}{u+1}\right|\right]_{\sqrt{2}}^2$$

$$= -\ln(9-6\sqrt{2}) \ \blacksquare$$

83. $$\int_1^2 \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \int_1^2 \frac{x^2+2x-1}{x(x+2)(2x-1)}dx = \int_1^2\left(-\frac{1}{10(x+2)}+\frac{1}{5(2x-1)}+\frac{1}{2x}\right)dx$$

$$= \left[-\frac{1}{10}\ln|x+2|+\frac{1}{10}\ln|2x-1|+\frac{1}{2}\ln|x|\right]_1^2 = \frac{3}{10}\ln 2+\frac{1}{5}\ln 3 = \frac{1}{10}\ln 72 \ \blacksquare$$

84. $$I=\int_0^{\frac{\pi}{3}} 13e^{2x}\cos 3x dx = \left[\frac{13}{2}e^{2x}\cos 3x\right]_0^{\frac{\pi}{3}} + \frac{3}{2}\int_0^{\frac{\pi}{3}} 13e^{2x}\sin 3x dx$$

$$= -\frac{13}{2}(e^{2\pi/3}+1)+\frac{39}{2}\left\{\left[\frac{1}{2}e^{2x}\sin 3x\right]_0^{\frac{\pi}{3}} - \frac{1}{2}\int_0^{\frac{\pi}{3}} 3e^{2x}\cos 3x dx\right\}$$

$$= -\frac{13}{2}(e^{2\pi/3}+1)-\frac{9}{4}I$$

$$\frac{13}{4}I = -\frac{13}{2}(e^{2\pi/3}+1)$$

$$\therefore I=\int_0^{\frac{\pi}{3}} 13e^{2x}\cos 3x dx = -2(e^{2\pi/3}+1) \ \blacksquare$$

85. $$\int_0^{\pi} e^{-x}\sin^2 2x dx = \frac{1}{2}\int_0^{\pi} e^{-x}(1-\cos 4x)dx = \frac{1}{2}[-e^{-x}]_0^{\pi} - \frac{1}{2}\int_0^{\pi} e^{-x}\cos 4x dx$$

$$= \frac{1-e^{-\pi}}{2} - \frac{1}{2}\left(\left[\frac{1}{4}e^{-x}\sin 4x\right]_0^{\pi} + \frac{1}{4}\int_0^{\pi} e^{-x}\sin 4x dx\right)$$

$$= \frac{1-e^{-\pi}}{2} - \frac{1}{8}\left(\left[-\frac{1}{4}e^{-x}\cos 4x\right]_0^{\pi} - \frac{1}{4}\int_0^{\pi} e^{-x}\cos 4x dx\right)$$

$$= \frac{15(1-e^{-\pi})}{32} + \frac{1}{32}\int_0^{\pi} e^{-x}\cos 4x dx$$

$$\frac{17}{32}\int_0^{\pi} e^{-x}\cos 4x dx = \frac{1-e^{-\pi}}{32}, \quad \int_0^{\pi} e^{-x}\cos 4x dx = \frac{1-e^{-\pi}}{17}$$

$$\therefore \int_0^{\pi} e^{-x}\sin^2 2x dx = \frac{1-e^{-\pi}}{2} - \frac{1}{2}\int_0^{\pi} e^{-x}\cos 4x dx = \frac{8}{17}(1-e^{-\pi}) \ \blacksquare$$

86. sol 1) $t^2 = \tan x$, $2tdt = \sec^2 x dx$, $dx = \dfrac{2t}{t^4+1}dt$

$$\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\, dx = \int_0^1 \frac{2t^2}{t^4+1}dt = \int_0^1 \frac{2}{t^2+\frac{1}{t^2}}dt = \int_0^1 \frac{\left(1+\frac{1}{t^2}\right)+\left(1-\frac{1}{t^2}\right)}{t^2+\frac{1}{t^2}}dt$$

$$= \int_0^1 \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt + \int_0^1 \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt = \int_0^1 \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{2})^2}dt + \int_0^1 \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-(\sqrt{2})^2}dt$$

$$= \left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-1/t}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{t+1/t-\sqrt{2}}{t+1/t+\sqrt{2}}\right|\right]_0^1 = \frac{\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}}\ln(\sqrt{2}-1) \ \blacksquare$$

sol 2) $I := \displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\, dx$, $J := \displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\cot x}\, dx$

$$I + J = \int_0^{\frac{\pi}{4}} (\sqrt{\tan x} + \sqrt{\cot x})dx = \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{(\sin x - \cos x)'}{\sqrt{1-(\sin x - \cos x)^2}}dx = \sqrt{2}\left[\sin^{-1}(\sin x - \cos x)\right]_0^{\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}} \quad \cdots \quad [1]$$

$$I-J=\int_0^{\frac{\pi}{4}}(\sqrt{\tan x}-\sqrt{\cot x})dx = \sqrt{2}\int_0^{\frac{\pi}{4}}\frac{\sin x-\cos x}{\sqrt{\sin 2x}}dx$$

$$= -\sqrt{2}\int_0^{\frac{\pi}{4}}\frac{(\sin x+\cos x)'}{\sqrt{(\sin x+\cos x)^2-1}}dx$$

$$= \left[-\sqrt{2}\ln\left|(\sin x+\cos x)+\sqrt{(\sin x+\cos x)^2-1}\right|\right]_0^{\frac{\pi}{4}}$$

$$= -\sqrt{2}\ln(\sqrt{2}+1) = \sqrt{2}\ln(\sqrt{2}-1)\ \cdots\ [2]$$

$$\frac{[1]+[2]}{2}\ :\ \therefore I=\int_0^{\frac{\pi}{4}}\sqrt{\tan x}\,dx = \frac{\pi}{2\sqrt{2}}+\frac{1}{\sqrt{2}}\ln(\sqrt{2}-1)\ \blacksquare$$

87. $t=\tan x,\ dt=\sec^2 x dx,\ u=\dfrac{t}{1-t},\ du=\dfrac{1}{(1-t)^2}dt,\ u=y^2,\ du=2ydy$

$$I=\int_0^{\frac{\pi}{4}}\sqrt{\tan x}\sqrt{1-\tan x}\,dx = \int_0^1\frac{\sqrt{t(1-t)}}{1+t^2}dt = \int_0^\infty\frac{\sqrt{u}}{(1+u)(1+2u+2u^2)}du$$

$$=\int_0^\infty\frac{2y^2}{(1+y^2)(1+2y^2+2y^4)}dy = 2\int_0^\infty\left(\frac{2y^2}{1+2y^2+2y^4}+\frac{1}{1+2y^2+2y^4}-\frac{1}{1+y^2}\right)dy$$

$$=\int_{-\infty}^\infty\left(\frac{2y^2}{1+2y^2+2y^4}+\frac{1}{1+2y^2+2y^4}-\frac{1}{1+y^2}\right)dy = I_1+I_2-\left[\tan^{-1}y\right]_{-\infty}^{\infty} = I_1+I_2-\pi$$

$$I_1=\int_{-\infty}^\infty\frac{2y^2}{1+2y^2+2y^4}dy = \int_{-\infty}^\infty\frac{1}{y^2+\dfrac{1}{2y^2}+1}dy = \int_{-\infty}^\infty\frac{1}{\left(y-\dfrac{1}{\sqrt{2}y}\right)^2+1+\sqrt{2}}dy$$

$$=\int_{-\infty}^\infty\frac{1}{y^2+1+\sqrt{2}}dy\ (\because\ \text{Glasser's Master Theorem})$$

$$=\left[\frac{1}{\sqrt{1+\sqrt{2}}}\tan^{-1}\left(\frac{y}{\sqrt{1+\sqrt{2}}}\right)\right]_{-\infty}^{\infty} = \frac{\pi}{\sqrt{1+\sqrt{2}}}$$

$$I_2=\frac{1}{2}\int_{-\infty}^\infty\frac{1}{y^4+y^2+\dfrac{1}{2}}dy = \int_0^\infty\frac{1}{y^2+\dfrac{1}{2y^2}+1}\cdot\frac{dy}{y^2} = \int_0^\infty\frac{1}{\dfrac{y^2}{2}+\dfrac{1}{y^2}+1}dy\ \left(y\mapsto\frac{1}{y}\right)$$

$$= \int_{-\infty}^{\infty} \frac{1}{y^2 + \frac{2}{y^2} + 2} dy = \int_{-\infty}^{\infty} \frac{1}{\left(y - \frac{\sqrt{2}}{y}\right)^2 + 2 + 2\sqrt{2}} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{y^2 + 2 + 2\sqrt{2}} dy \ (\because \text{ Glasser's Master Theorem})$$

$$= \left[\frac{1}{\sqrt{2+2\sqrt{2}}} \tan^{-1}\left(\frac{y}{\sqrt{2+2\sqrt{2}}} \right) \right]_{-\infty}^{\infty} = \frac{\pi}{\sqrt{2+2\sqrt{2}}}$$

$$\therefore I = \int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sqrt{1 - \tan x}\, dx = I_1 + I_2 - \pi = \left(\sqrt{\frac{1+\sqrt{2}}{2}} - 1 \right)\pi \ \blacksquare$$

88. $I_{m,n} := \displaystyle\int_1^{\infty} \frac{(\ln x)^m}{x^n} dx = \int_1^{\infty} \frac{(\ln x)^m}{x} \cdot \frac{1}{x^{n-1}} dx$

$$= \left[\frac{(\ln x)^{m+1}}{(m+1)x^{n-1}} \right]_1^{\infty} - \frac{1-n}{m+1} \int_1^{\infty} \frac{(\ln x)^{m+1}}{x^n} dx = \frac{n-1}{m+1} \int_1^{\infty} \frac{(\ln x)^{m+1}}{x^n} dx$$

$= \dfrac{n-1}{m+1} I_{m+1,n}$, $\quad I_{m,n} = \dfrac{m}{n-1} I_{m-1,n}$을 얻는다. 한편

$I_{0,n} = \displaystyle\int_1^{\infty} \frac{1}{x^n} dx = \frac{1}{n-1}$이므로

$I_{m,n} = \dfrac{m!}{(n-1)^{m+1}}$, $m, n \in \mathbb{N}_0$.

$$\therefore \int_1^{\infty} \frac{(\ln x)^{627}}{x^{2022}} dx = I_{627,2022} = \frac{627!}{2021^{628}} \ \blacksquare$$

89. $x = \cot t$, $\sin(\cot^{-1} x) = \sin t = \dfrac{x}{\sqrt{1+x^2}}$, $x = \tan u$, $\cos(\tan^{-1} x) = \cos u = \dfrac{1}{\sqrt{1+x^2}}$

$$\int_0^{\sqrt{6}} \cos^2(\tan^{-1}(\sin(\cot^{-1} x)))dx = \int_0^{\sqrt{6}} \cos^2\left(\tan^{-1}\left(\frac{x}{\sqrt{1+x^2}} \right) \right) dx$$

$$= \int_0^{\sqrt{6}} \frac{1+x^2}{2+x^2} dx = \int_0^{\sqrt{6}} \left(1 - \frac{1}{2+x^2} \right) dx = \left[x - \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}} \right) \right]_0^{\sqrt{6}} = \sqrt{6} - \frac{\pi}{3\sqrt{2}} \ \blacksquare$$

90. $t=\tan x$, $dt=\sec^2 x dx$, $u=t+\sqrt{1+t^2}$, $t=\frac{1}{2}\left(u-\frac{1}{u}\right)$, $dt=\frac{1}{2}\left(1+\frac{1}{u^2}\right)du$

$$\int_0^{\frac{\pi}{6}}\frac{\sec^2 x}{(\sec x+\tan x)^{5/2}}dx=\int_0^{\frac{1}{\sqrt{3}}}\frac{1}{(t+\sqrt{1+t^2})^{5/2}}dt=\frac{1}{2}\int_1^{\sqrt{3}}(u^{-5/2}+u^{-9/2})du$$

$$=\left[-\frac{1}{3}u^{-3/2}-\frac{1}{7}u^{-7/2}\right]_1^{\sqrt{3}}=-\frac{8}{63}\sqrt[4]{3}+\frac{10}{21}\ \blacksquare$$

91. 제 1종 오일러 치환 : $\sqrt{x^2+4x-4}=x+t$, $x=\frac{t^2+4}{4-2t}$, $dx=\frac{-2t^2+8t+8}{(4-2t)^2}dt$

$$\int_1^2\frac{1}{x\sqrt{x^2+4x-4}}dx=\int_0^{2\sqrt{2}-2}\frac{4-2t}{t^2+4}\cdot\frac{4-2t}{-t^2+4t+4}\cdot\frac{-2t^2+8t+8}{(4-2t)^2}dt$$

$$=2\int_0^{2\sqrt{2}-2}\frac{1}{t^2+4}dt=\left[\tan^{-1}\left(\frac{t}{2}\right)\right]_0^{2\sqrt{2}-2}=\tan^{-1}(\sqrt{2}-1)=\frac{\pi}{8}\ \blacksquare$$

92. 제 2종 오일러 치환 : $\sqrt{-x^2+x+2}=xt+\sqrt{2}$, $x=\frac{1-2\sqrt{2}t}{t^2+1}$, $dx=\frac{2\sqrt{2}t^2-2t-2\sqrt{2}}{(t^2+1)^2}dt$

$$\int_1^{\frac{25}{73}}\frac{1}{x\sqrt{-x^2+x+2}}dx$$

$$=\int_0^{\frac{4\sqrt{2}}{25}}\frac{t^2+1}{1-2\sqrt{2}t}\cdot\frac{t^2+1}{-\sqrt{2}t^2+t+\sqrt{2}}\cdot\frac{2\sqrt{2}t^2-2t-2\sqrt{2}}{(t^2+1)^2}dt$$

$$=\int_0^{\frac{4\sqrt{2}}{25}}\frac{-2}{-2\sqrt{2}t+1}dt=\frac{1}{\sqrt{2}}\int_0^{\frac{4\sqrt{2}}{25}}\frac{-2\sqrt{2}}{-2\sqrt{2}t+1}dt=\left[\frac{1}{\sqrt{2}}\ln\left|2\sqrt{2}t-1\right|\right]_0^{\frac{4\sqrt{2}}{25}}$$

$$=\sqrt{2}\ln\frac{3}{5}\ \blacksquare$$

93. 제 3종 오일러 치환 : $\sqrt{-(x-2)(x-1)}=(x-2)t$, $x=\frac{-2t^2-1}{-t^2-1}$, $dx=\frac{2t}{(-t^2-1)^2}dt$

$$\int_1^2\frac{x^2}{\sqrt{-x^2+3x-2}}dx=\int_0^{-\infty}\left(\frac{-2t^2-1}{-t^2-1}\right)^2\cdot\frac{-t^2-1}{t}\cdot\frac{2t}{(-t^2-1)^2}dt$$

$$= -2\int_0^{-\infty}\frac{(2t^2+1)^2}{(t^2+1)^3}dt = -2\int_0^{-\infty}\left(\frac{4}{t^2+1}-\frac{4}{(t^2+1)^2}+\frac{1}{(t^2+1)^3}\right)dt$$

$$= \left[-8\tan^{-1}t+8I(t)-2J(t)\right]_0^{-\infty} = \left[-\frac{19}{4}\tan^{-1}t+\frac{t(13t^2+11)}{4(t^2+1)^2}\right]_0^{-\infty} = \frac{19}{8}\pi \ ■$$

※ $t=\tan u$, $dt=\sec^2 u du$, $I(t)=\int_0^{-\infty}\frac{1}{(t^2+1)^2}dt=\int_0^{-\frac{\pi}{2}}\cos^2 u du$

$$= \frac{1}{2}\int_0^{-\frac{\pi}{2}}(1+\cos 2u)du = \left[\frac{1}{2}u+\frac{1}{4}\sin 2u\right]_0^{-\frac{\pi}{2}} = \left[\frac{1}{2}\tan^{-1}t+\frac{t}{2(1+t^2)}\right]_0^{-\infty}$$

※※ $t=\tan u$, $dt=\sec^2 u du$, $J(t)=\int_0^{-\infty}\frac{1}{(t^2+1)^3}dt=\int_0^{-\frac{\pi}{2}}\cos^4 u du$

$$= \frac{1}{4}\int_0^{-\frac{\pi}{2}}(1+\cos 2u)^2 du = \frac{1}{4}\int_0^{-\frac{\pi}{2}}(1+2\cos 2u+\cos^2 2u)du$$

$$= \frac{1}{4}\int_0^{-\frac{\pi}{2}}\left(1+2\cos 2u+\frac{1+\cos 4u}{2}\right)du = \frac{1}{8}\int_0^{-\frac{\pi}{2}}(3+4\cos 2u+\cos 4u)du$$

$$= \left[\frac{3}{8}u+\frac{1}{4}\sin 2u+\frac{1}{32}\sin 4u\right]_0^{-\frac{\pi}{2}} = \left[\frac{3}{8}\tan^{-1}t+\frac{t(3t^2+5)}{8(t^2+1)^2}\right]_0^{-\infty}$$

94. $t=\cos\theta$, $dt=-\sin\theta d\theta$, $u=\frac{t-1}{t+1}$, $t=\frac{1+u}{1-u}$, $dt=\frac{2}{(1-u)^2}du$

$$\int_0^{\frac{\pi}{4}}\frac{\sin^3(\theta/2)}{\cos(\theta/2)\cdot\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}}d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{(1-\cos\theta)\sin\theta}{(1+\cos\theta)\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}}d\theta = \frac{1}{2}\int_1^{\frac{1}{\sqrt{2}}}\frac{t-1}{(t+1)\sqrt{t^3+t^2+t}}dt$$

$$= \frac{1}{2}\int_0^{2\sqrt{2}-3}u\cdot\frac{\sqrt{(1-u)^3}}{\sqrt{(1+u)^3+(1+u)^2(1-u)+(1+u)(1-u)^2}}\cdot\frac{2}{(1-u)^2}du$$

$$= \int_0^{2\sqrt{2}-3}\frac{u}{\sqrt{3-2u^2-u^4}}du = \frac{1}{2}\int_0^{2\sqrt{2}-3}\frac{2u}{\sqrt{4-(u^2+1)^2}}du$$

$$= \left[\frac{1}{2}\sin^{-1}\left(\frac{u^2+1}{2}\right)\right]_0^{2\sqrt{2}-3} = \frac{1}{2}\sin^{-1}(9-6\sqrt{2}) - \frac{\pi}{12} \ ■$$

95. $$\int_0^{\frac{\pi}{6}} \frac{\tan^4\theta}{1-\tan^2\theta}d\theta = \int_0^{\frac{\pi}{6}} \frac{\tan^4\theta - 1}{1-\tan^2\theta}d\theta + \int_0^{\frac{\pi}{6}} \frac{1}{1-\tan^2\theta}d\theta$$

$$= -\int_0^{\frac{\pi}{6}} (\tan^2\theta+1)d\theta + \int_0^{\frac{\pi}{6}} \frac{\cos^2\theta}{\cos^2\theta-\sin^2\theta}d\theta = -\int_0^{\frac{\pi}{6}} \sec^2\theta d\theta + \int_0^{\frac{\pi}{6}} \frac{1+\cos 2\theta}{2\cos 2\theta}d\theta$$

$$= [-\tan\theta]_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} \left(\frac{1}{2}\sec 2\theta + \frac{1}{2}\right)d\theta = \left[-\tan\theta + \frac{1}{4}\ln|\sec 2\theta + \tan 2\theta| + \frac{1}{2}\theta\right]_0^{\frac{\pi}{6}}$$

$$= -\frac{1}{\sqrt{3}} + \frac{\pi}{12} + \frac{1}{4}\ln(2+\sqrt{3}) \ ■$$

96. $$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}dx = \int_{-\infty}^{\infty} \frac{1}{\left(x-\frac{1}{x}\right)^2+2}dx = \int_{-\infty}^{\infty} \frac{1}{x^2+2}dx$$ ($\because$ Glasser's Master Theorem)

$$= \left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_{-\infty}^{\infty} = \frac{\pi}{\sqrt{2}} \ ■$$

97. $$\int_0^{\infty} \exp\left(-x^2-\frac{1}{x^2}\right)dx = \int_0^{\infty} \exp\left[-\left(x-\frac{1}{x}\right)^2-2\right]dx = \frac{1}{2e^2}\int_{-\infty}^{\infty} \exp\left[-\left(x-\frac{1}{x}\right)^2\right]dx$$

$$= \frac{1}{2e^2}\int_{-\infty}^{\infty} e^{-u^2}du = \frac{\sqrt{\pi}}{2e^2} \ ■$$ ($\because$ Glasser's Master Theorem)

98. $$\int_0^1 \frac{1-x^{99}}{(1+x)(1+x^{100})}dx = \int_0^1 \frac{(1+x^{100})-x^{99}(1+x)}{(1+x)(1+x^{100})}dx = \int_0^1 \left(\frac{1}{1+x} - \frac{x^{99}}{1+x^{100}}\right)dx$$

$$= \left[\ln|1+x| - \frac{1}{100}\ln|1+x^{100}|\right]_0^1 = \frac{99}{100}\ln 2 \ ■$$

99. $$\int_{-\infty}^{\infty} \exp\left(-\frac{(x^2-13x-1)^2}{611x^2}\right)dx = \int_{-\infty}^{\infty} \exp\left(-\frac{(x-x^{-1}-13)^2}{611}\right)dx$$

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{(x-13)^2}{611}\right)dx$$ ($\because$ Glasser's Master Theorem)

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{611}\right) dx \ \ (x \mapsto x+13,\ dx \mapsto dx)$$

$$= \int_{-\infty}^{\infty} \exp(-u^2) \cdot \sqrt{611}\, du \ \ \left(u = \frac{x}{\sqrt{611}},\ dx = \sqrt{611}\, du\right)$$

$$= \sqrt{611\pi} \ \blacksquare$$

100. $t = \tan x,\ dt = \sec^2 x dx$

$\int_0^{\frac{\pi}{2}} \sqrt[n]{\tan x}\, dx = \int_0^{\infty} \frac{t^{\frac{1}{n}}}{1+t^2} dt,\quad \int_0^{\infty} \frac{x^a}{1+x^2} dx = \frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right)$임을 이용하자.

sol 1) $t = e^x,\ dt = e^x dx$

$$\int_0^{\infty} \frac{x^a}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{e^{(a+1)t}}{1+e^{2t}} dt = \int_{-\infty}^{0} \frac{e^{(a+1)t}}{1+e^{2t}} dt + \int_0^{\infty} \frac{e^{(a-1)t}}{1+e^{-2t}} dt$$

한편 $\frac{e^{(a+1)t}}{1-(-e^{2t})}$, $\frac{e^{(a-1)t}}{1-(-e^{-2t})}$는 공비가 각각 $(-e^{2t})$, $(-e^{-2t})$이고 초항이 각각 $e^{(a+1)t}$, $e^{(a-1)t}$인 등비급수의 합이므로 이를 역으로 전개할 수 있다.

$$\int_{-\infty}^{0} \frac{e^{(a+1)t}}{1+e^{2t}} dt + \int_0^{\infty} \frac{e^{(a-1)t}}{1+e^{-2t}} dt$$

$$= \int_{-\infty}^{0} \sum_{n=0}^{\infty} e^{(a+1)t} \cdot (-e^{2t})^n dt + \int_0^{\infty} \sum_{n=0}^{\infty} e^{(a-1)t} \cdot (-e^{-2t})^n dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\int_{-\infty}^{0} e^{(2n+1+a)t} dt + \int_0^{\infty} e^{-(2n+1-a)} dt\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1+a} + \frac{1}{2n+1-a}\right) = 2\sum_{n=0}^{\infty} (-1)^n \left(\frac{2n+1}{(2n+1)^2 - a^2}\right) \ \cdots \ [\#]$$

한편 $\sec(z)$의 Mittag-Leffler 전개식은

$$\sec(z) = \pi \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(\pi/2)^2 (2n+1)^2 - z^2}$$

이므로 (참고 : https://cdlbb.github.io/WandW/CMA07-3-FactorTheoremMN.html#7.4theexpansionofaclassoffunctionsinrationalfractions.)

$z=\frac{a\pi}{2}$를 대입하고 $\frac{\pi}{2}$를 곱하면

$$\frac{\pi}{2}\sec\left(\frac{a\pi}{2}\right)=\frac{\pi^2}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(\pi/2)^2(2n+1)^2-(a\pi/2)^2}=2\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)^2-a^2}$$

을 얻고 이는 [#]의 전개식과 일치한다.

$$\therefore \int_0^{\infty}\frac{x^a}{1+x^2}dx=\frac{\pi}{2}\sec\left(\frac{a\pi}{2}\right) \blacksquare$$

sol 2) $t=x^2$, $dt=2xdx$

$$\int_0^{\infty}\frac{x^a}{1+x^2}dx=\frac{1}{2}\int_0^{\infty}\frac{t^{(a-1)/2}}{1+t}dt=\frac{1}{2}B\left(\frac{a+1}{2},1-\frac{a+1}{2}\right)$$

$$=\frac{1}{2}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(1-\frac{a+1}{2}\right)=\frac{\pi}{2}\csc\left(\frac{(a+1)\pi}{2}\right)=\frac{\pi}{2}\sec\left(\frac{a\pi}{2}\right) \blacksquare$$

※ 베타함수 $B(a,b)$는 다음과 같이 정의되는 이변수 함수이다.

$$B(a,b):=\int_0^{\infty}\frac{z^{a-1}}{1+z^{a+b}}dz$$

또한 이는 다음과 같이 감마함수의 비로 표현할 수 있다.

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

감마함수 $\Gamma(z)$는 다음과 같이 정의되며, 이는 계승 함수($n!$)의 정의역을 실수부가 양수인 복소수 전체로 확장(음의 정수는 제외)한 것이다. (29번 문제의 1번 솔루션은 이 감마함수의 정의를 이용한 것으로, 자연수 n에 대하여 $\Gamma(n)=(n-1)!$이다.) 감마함수는 계승 함수를 해석적으로 확장한 것이므로 팩토리얼의 성질이 그대로 성립하여,
$\Gamma(x+1)=x\Gamma(x)$이고 $\Gamma(1)=1$이다.)

$$\Gamma(z):=\int_0^{\infty}t^{z-1}e^{-t}dt \ \ (Re(z)>0)$$

감마함수에 대하여 다음과 같은 오일러 반사 공식이 성립하며, 상술한 2번 솔루션의 마지

막 단계에서 이 공식이 사용되었다.

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$$

위 공식은 Residue Theorem을 사용하여 직접 복소적분을 하거나 감마함수의 정의식을 대입하여 증명할 수 있으며, 본문에서는 복소적분을 사용하지 않는다는 철학을 유지하기 위해 다른 증명법을 제시한다. 복소적분을 포함한 다른 증명법에 관심이 있는 독자는 아래 링크의 증명들을 참고하기 바란다.

https://math.stackexchange.com/questions/714482/prove-that-gammap-times-gamma1-p-frac-pi-sin-p-pi-forall-p-in?noredirect=1&lq=1

pf) 감마함수의 바이어슈트라스 무한곱 꼴을 생각하자. (감마함수는 실제로 여러 가지 형태로 정의되고 이들은 모두 동치이다. 상술한 정의는 적분 형태이다.)

$$\frac{1}{\Gamma(p)} = pe^{\gamma p}\prod_{n=1}^{\infty}\left(1+\frac{p}{n}\right)e^{-\frac{p}{n}}$$

$$\frac{1}{\Gamma(p)}\cdot\frac{1}{\Gamma(-p)} = pe^{\gamma p}\prod_{n=1}^{\infty}\left(1+\frac{p}{n}\right)e^{-\frac{p}{n}}\cdot(-p)e^{-\gamma p}\prod_{n=1}^{\infty}\left(1-\frac{p}{n}\right)e^{\frac{p}{n}} = -p^2\prod_{n=1}^{\infty}\left(1-\frac{p^2}{n^2}\right)$$

한편 감마함수의 성질에 의해 $\Gamma(1-p) = -p\Gamma(-p)$이므로

$$\frac{1}{\Gamma(1-p)\Gamma(p)} = p\prod_{n=1}^{\infty}\left(1-\frac{p^2}{n^2}\right)$$

이고 바이어슈트라스 분해 정리에 의해

$$\sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$$

임을 이용하면

$$\frac{1}{\Gamma(1-p)\Gamma(p)} = p\prod_{n=1}^{\infty}\left(1-\frac{p^2}{n^2}\right) = \frac{\sin(\pi p)}{\pi}$$

얻고 증명이 완료되었다. ■

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt[n]{\tan x}\,dx = \int_0^{\infty} \frac{t^{\frac{1}{n}}}{1+t^2}\,dt = \frac{\pi}{2}\sec\left(\frac{\pi}{2n}\right) \blacksquare \ (2 \le n \in \mathbb{N})$$